Response to the discussion by Zhang and Lytton on the paper, "a new modeling approach for unsaturated soils using independent stress variables", by Daichao Sheng, Delwyn G. Fredlund and Antonio Gens, April 2008, Vol. 45, pp. 511-534.

Response by:

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We would like to thank the discussers for their interest in our paper and also for their challenging questions that have shown their thorough knowledge on the subject. Their comments have enhanced our understanding of the subject area and hopefully may lead to further constitutive model developments.

Question 1: Stress-Path Dependency within the Elastic Zone

The stress-path dependent elastic response will lead to hysteretic behaviour. As such, a loading path A-B-D followed by an unloading path D-C-A in Figure 1 of the discussion would not lead to zero volumetric strain. This hysteretic behaviour constitutes a restriction in classical elastoplasticity theory. We have carefully considered all possible options in the literature and find that the SFG model still provides a better solution than most other options.

It should be noted that the equation derived in the discussion for stress path ABD has a small misprint and the correct equation should read:

[1] $$\Delta\left(\ln v\right)\Big|_{\mathrm{ABD}} = -\kappa_{\mathrm{vp}} \ln\frac{p+s}{p_0+s} + \frac{\kappa_{\mathrm{vp}}\left(s_{\mathrm{sa}}+1\right)}{p_0-1}\left[\ln\frac{p_0+s}{s+1} - \ln\frac{p_0+s_0}{s_0+1}\right]$$

The volume change for stress path ACD is correctly derived:

[2] $$\Delta\left(\ln v\right)\Big|_{\mathrm{ACD}} = -\kappa_{\mathrm{vp}} \ln\frac{p+s_0}{p_0+s_0} + \frac{\kappa_{\mathrm{vp}}\left(s_{\mathrm{sa}}+1\right)}{p-1}\left[\ln\frac{p+s}{s+1} - \ln\frac{p+s_0}{s_0+1}\right]$$

Equation [1] is indeed different from [2], but their actual difference is very small. A numerical example is given below. Some qualitative remarks are warranted. When suction is

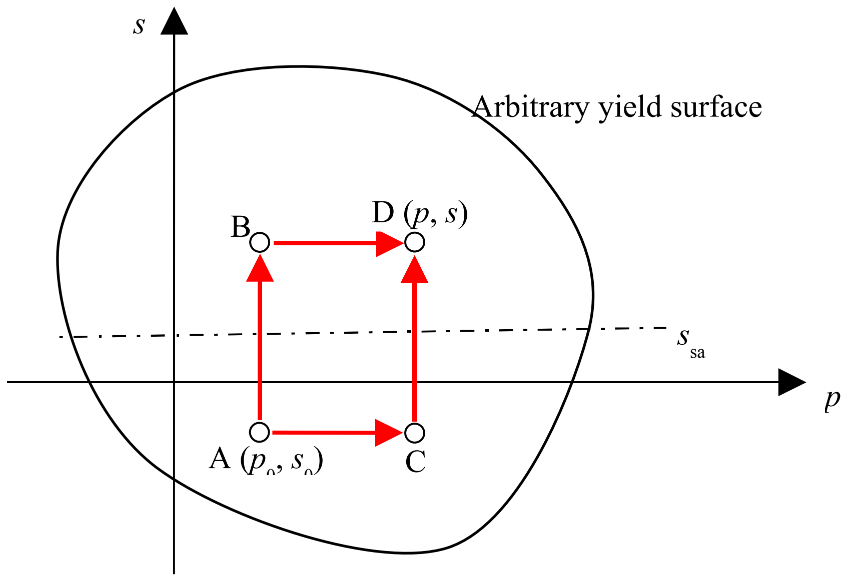

Figure 1. Stress paths within the elastic zone.

below the saturation suction, the SFG model is stress-path independent. When suction is above but close to the saturation suction, κ_{vs} is close to κ_{vp}, and hence the volume change difference between stress path ABD and ACD is small. On the other hand, when suction is very large, the volume change ($-\kappa_{vp}\dfrac{dp}{p+s}$) caused by a stress change is close to zero (because of the term ($p + s$)) and the volume change caused by a suction change is also close to zero (because of κ_{vs}). Therefore, the difference in the volume changes between the two stress paths is also small.

Option 1: Other models in terms of net stress

Let us now give consideration to other alternatives. The following elastic model is commonly used in the literature (also used in Zhang and Lytton, 2007):

[3] $$dv^e = -\kappa_{vp}\frac{dp}{p} - \kappa_{vs}\frac{ds}{s}$$

To eliminate singularity when s = 0, the above equation is sometimes replaced by the following equation:

[4] $$dv^{e} = -\kappa_{vp}\frac{dp}{p} - \kappa_{vs}\frac{ds}{s + p_{at}}$$

where p_{at} is atmospheric pressure.

Equations [3] and [4] appear to be stress-path independent at first glance, at least for constant of κ_{vs} and κ_{vp} values. However, a closer examination reveals that these equations give inconsistent volume changes and are also stress-path dependent whenever the stress state changes from saturated to unsaturated states or *vice versa*. Let us consider stress path ABD and ACD in Figure 1. For simplicity, let us assume that the pore air pressure remains atmospheric. In this case, suction is equivalent to the negative pore water pressure. Let us also assume that $\kappa_{vs} = \kappa_{vp}$ and the saturation suction is constant. For saturated soils, the effective stress principle holds and the elastic volume change is given by:

[5] $$dv^{e} = -\kappa_{vp}\frac{d(p + s)}{p + s} = -\kappa_{vp}\frac{dp}{p + s} - \kappa_{vp}\frac{ds}{p + s}$$

The volume change for a suction change from A to B can then be written:

[6] $$\Delta v^{e}\big|_{AB} = -\kappa_{vp}\ln\frac{p_0 + s_{sa}}{p_0 + s_0} - \kappa_{vp}\ln\frac{s}{s_{sa}} = -\kappa_{vp}\ln\left(\frac{p_0 + s_{sa}}{p_0 + s_0}\frac{s}{s_{sa}}\right)$$

The volume change from C to D is:

[7] $$\Delta v^{e}\big|_{CD} = -\kappa_{vp}\ln\frac{p + s_{sa}}{p + s_0} - \kappa_{vp}\ln\frac{s}{s_{sa}} = -\kappa_{vp}\ln\left(\frac{p + s_{sa}}{p + s_0}\frac{s}{s_{sa}}\right)$$

The volume change from B to D is:

[8] $$\Delta v^{e}\big|_{BD} = -\kappa_{vp}\ln\frac{p}{p_0}$$

The volume change from A to C is:

[9] $$\Delta v^{e}\Big|_{AC} = -\kappa_{vp}\ln\frac{p+s_0}{p_0+s_0}$$

It can be seen that the elastic volume changes along ABD and ACD can be quite different. The elastic volume changes are stress-path independent only when points A, B, C and D are all in the saturated or unsaturated zone. Equation [4] will lead to even stronger stress path dependency due to the parameter, p_{at}. It also makes the volume change due to a suction change insignificant when s is smaller than p_{at}.

Let us consider a numerical example where,

$$s_{sa} = 10 \text{ kPa},\ p_0 = 11 \text{ kPa},\ s_0 = -10 \text{ kPa},\ p = 20 \text{ kPa},\ s = 20 \text{ kPa}$$

The total volume change when following stress path ABD is:

$$\Delta v^{e}\Big|_{ABD} = -\kappa_{vp}\ln\left(\frac{p_0+s_{sa}}{p_0+s_0}\frac{s}{s_{sa}}\right) - \kappa_{vp}\ln\frac{p}{p_0} = -\kappa_{vp}\ln\left(\frac{21}{1}\frac{20}{10}\frac{20}{11}\right) = -\kappa_{vp}\ln(76.4)$$

The total volume change over stress path ACD is:

$$\Delta v^{e}\Big|_{ACD} = -\kappa_{vp}\ln\left(\frac{p+s_{sa}}{p+s_0}\frac{s}{s_{sa}}\right) - \kappa_{vp}\ln\frac{p+s_0}{p_0+s_0} = -\kappa_{vp}\ln\left(\frac{30}{10}\frac{20}{10}\frac{10}{1}\right) = -\kappa_{vp}\ln(60)$$

The volume change difference between the two stress paths is then $\left(\kappa_{vp}\ln\frac{76.4}{60} = \kappa_{vp}\ln 1.27\right)$, or about 6% of the predicted total volume change. On the other hand, the SFG model would predict a volume change difference of $\left(\kappa_{vp}\ln\frac{38.16}{36.85} = \kappa_{vp}\ln 1.03\right)$, or about 0.8% of the total volume change[1]. Neither of these numbers is very significant, considering the relatively small value of κ_{vp} (typically in the order of 0.01). Equations [3] or [4] are clearly not a

[1] To obtain this figure, we have to integrate the following SFG equation along the stress paths:

$$\mathrm{d}v = -\kappa_{vp}\frac{\mathrm{d}p}{p+s} - \kappa_{vs}\frac{\mathrm{d}s}{p+s}$$

A semi-logarithmic SFG relationship is used in order to make the parameter κ_{vp} comparable with that used in equation (3). The volume changes along stress paths ABD and ACD are respectively:

better option than that proposed in the SFG model in terms of stress-path dependency.

A more serious problem that arises with the use of equations [3] and [4] is that the volume change caused by a stress change becomes undefined when points A and B (or C and D) have the same suction, but A is inside the saturated zone and B is inside the unsaturated zone. This inconsistence comes from the fact that neither equation [3] or [4] recovers equation [5] when an unsaturated soil becomes saturated. As a consequence, the stiffness matrix in the incremental stress-strain relationship becomes undefined at the transition suction between saturated and unsaturated states.

Option 2: Constant κ_{vs}

Let us now consider the usage of a constant κ_{vs} and let it be set to κ_{vp}. This alternative has been used by some models and it leads to stress-path independency within the elastic zone. This approach could also have been adopted for the SFG model. The only difference to the model would then be the integration of equation [11] in Sheng *et al.* (2008). The equation would still be integrable and would result in a somewhat different yield stress p_{y}. However, this alternative would not agree with the fundamental fact that changing suction under constant stress does not cause much volume change when suction is very high. This is the reason that parameter λ_{vs} must approach zero as the suction approaches 1,000,000 kPa (or even residual suction). Obviously we cannot have a zero λ_{vs}, but a finite κ_{vs}. Such an alternative would cause a more significant problem than that of slight stress-path dependency.

Option 3: Integration according to Green Theorem

We could integrate the following equation as suggested by the discussers:

$$\Delta v^{\mathrm{e}}\Big|_{\mathrm{ABD}} = -\kappa_{\mathrm{vp}} \ln\left[\frac{p_0 + s_{\mathrm{sa}}}{p_0 + s_0}\right] - \kappa_{\mathrm{vp}} \frac{s_{\mathrm{sa}} + 1}{p_0 - 1} \ln\left[\frac{s + 1}{s_{\mathrm{sa}} + 1}\frac{p_0 + s_{\mathrm{sa}}}{p_0 + s}\right] - \kappa_{\mathrm{vp}} \ln\frac{p + s}{p_0 + s} = -\kappa_{\mathrm{vp}} \ln(36.85)$$

$$\Delta v^{\mathrm{e}}\Big|_{\mathrm{ACD}} = -\kappa_{\mathrm{vp}} \ln\left[\frac{p + s_0}{p_0 + s_0}\right] - \kappa_{\mathrm{vp}} \ln\frac{p + s_{\mathrm{sa}}}{p + s_0} - \kappa_{\mathrm{vp}} \frac{s_{\mathrm{sa}} + 1}{p - 1} \ln\left[\frac{s + 1}{s_{\mathrm{sa}} + 1}\frac{p + s_{\mathrm{sa}}}{p + s}\right] = -\kappa_{\mathrm{vp}} \ln(38.16)$$

Note that equations (1) and (2) cannot be used as s_0 is less than s_{sa}.

[10] $$\frac{\mathrm{d}\left(\frac{\kappa_{\mathrm{vp}}}{p+s}\right)}{\mathrm{d}s}-\frac{\mathrm{d}\left(\frac{\kappa_{\mathrm{vs}}}{p+s}\right)}{\mathrm{d}p}=0$$

to find a stress-path independent elastic model. The resulting κ_{vs} from such an integration would likely be a complex function of stress and suction. This function would likely be inconsistent with the basic feature of κ_{vs} (i.e. it must vary between κ_{vp} and zero as suction increases).

Option 4: Models in terms of effective stress

Alternatively, we could use the so-called effective stress approach:

[11] $$\mathrm{d}v^{\mathrm{e}}=-\kappa_{\mathrm{vp}}(s)\frac{\mathrm{d}p'}{p'}$$

This approach would obviously lead to a stress-path independent response in the effective stress space. However, before we start with this approach, we should first define what is meant by effective stress. Currently in the literature, all effective stress definitions for unsaturated soils involve material state variables (e.g. the degree of saturation) and can even depend on the stress path (e.g. the transition suction between saturated and unsaturated states). Therefore, the effective stress space where a constitutive relation is established is constantly changing with the material state or stress path. The resulting question is: what is the meaning of such a constitutive relationship?

Because the so-called effective stress definition for unsaturated soils usually depends on material state or even on stress path, it is impossible to control the stress or stress path in laboratory tests. It also presents difficulties in interpreting experimental results that are obtained under prescribed net stress and suction paths.

There are also constraints on the effective stress definition that have not been thoroughly discussed in the literature. For example, the effective mean stress must decrease slower than the yield stress as suction decreases and total mean stress is kept constant, in order to model wetting-induced collapse. On the other hand, the effective mean stress must increase faster than the yield stress as suction increases under constant total mean, in order to simulate drying-induced yielding of a slurry soil. Such constraints are not always consistent with one another.

We can always transfer, however, a known constitutive law established in the independent stress space into an effective stress space. Such a transfer is particularly relevant if it simplifies the model, removes its singularities and facilitates its implementation. For example, Sheng *et al*. (2003a, 2003b) have transferred the Barcelona Basic model (Alonso *et al.* 1990) into the Bishop effective stress space:

[12] $$\sigma'_{ij} = \sigma_{ij} - u_{\mathrm{a}}\delta_{ij} + \chi\left(u_{\mathrm{a}} - u_{\mathrm{w}}\right)\delta_{ij}$$

Certain assumptions usually have to be made during the transformation. For example, Sheng *et al.* (2003a, 2003b) used both S_{r} (degree of saturation) and $\sqrt{S_{\mathrm{r}}}$ for the parameter χ, in order to overcome the above-mentioned constraints. Further research is needed to transfer the SFG model into a so-called effective stress space.

In summarizing this question, the hysteretic behaviour within the elastic zone seems to be a feature of the models that use the independent stress variables. The SFG model seems to provide one of the better solutions to the problem raised by the discussers. There may be some space for improvement. It was also stated in our paper that the equation we used was not necessarily the one that must be used.

Question 2: Stress-Path Dependency of Elastoplastic part of SFG model

The classical critical state models (i.e. the original and the modified Cam clay models) are stress-path independent, in a sense that the plastic volumetric strain for a normally consolidated soil depends only on the starting and ending stress states. However, there are numerous data supporting stress-path dependent elastoplastic behaviour (e.g. Lade and Duncan, 1976; Nakai and Mihara, 1984). A number of elastoplastic models have therefore been developed to tackle stress-path dependent behaviour (e.g. Nakai, 1989; Kikumoto *et al.* 2007; Yao *et al.* 2008). In addition, most hypoplastic models (e.g. Kolymbas, 1991) and bounding surface models (e.g. Dafalias, 1986; Asaoka, 2003) can deal with stress-path dependency.

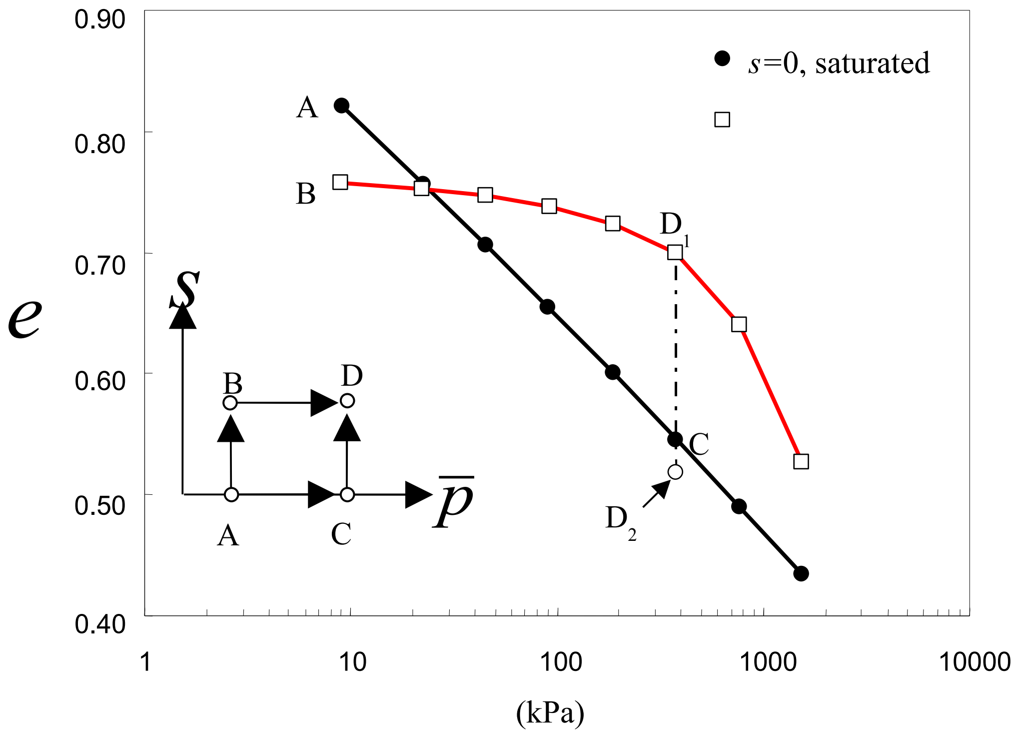

Figure 2. Oedometer tests on an air-dry silt soil by Jennings and Burland (1962).

For unsaturated soils, it is relatively easy to demonstrate the stress-path dependency. Jennings and Burland (1962) presented a set of oedometer test results on air-dry slurry soil and one of their plots is shown in Figure 2. Let us consider two stress paths: ABD and ACD, as shown in Figure 2. The soil at point A is on the normal compression line for saturated states, and the stress state must be on the yield surface. Therefore, drying the soil from point A to point B and then loading it to point D will result in an elastoplastic stress path. The void ratio will change from e_A to e_B and then to e_{D1}, with e_B and e_{D1} on the normal compression line for $S_r = 49.4\%$. Alternatively, compressing the soil from point A to point C and then drying it from point C to point D will also result in an elastoplastic stress path. The corresponding void ratio changes will follow the path ACD_2, with e_A and e_C on the normal compression line for $s = 0$ and e_{D2} below e_C. Point D_2 is estimated, because drying the soil from C to D will generally cause some volume decrease. It is then clear that the two stress paths lead to quite different volumetric responses. More recently Cunningham *et al.* (2003) presented similar results for air-dried unsaturated soils.

The crucial point here is to realize that points A, B, C, D are all on the current yield surfaces. A slurry soil that is air-dried and loaded according to path ABD responds in an elastoplastic manner, not just in an elastic manner. It is relatively easy to demonstrate the yield stress variation with suction for an air-dry unsaturated soil. Let us assume that the slurry soil was isotropically consolidated to Point C in Figure 3 and it has an air entry indicated by the suction at Point B^2. Because of the effective stress principle for saturated soils, the initial elastic zone is then bounded by the two thick lines that go through points A and C^1 and are inclined to horizontal by 45°. The yield (preconsolidation) stress at a suction corresponding to Point B^1 is then zero. Drying this slurry soil under zero mean stress to a suction represented by Point B^2 will then cause plastic yielding. The yield stress at zero suction will increase to the mean stress at Point C^2, but the preconsolidation stress at the current suction remains zero (Point B^2). Further drying will cause desaturation of the soil, but the yield stress at zero suction will generally not increase at the same rate as the suction. Let us dry the soil under zero stress to the suction at Point B^3. The yield stress at zero suction will then be somewhere between Point C^2 and Point C^4 (point C^3 in Figure 3). Let us now isotropically compress the soil under the constant suction at Point B^3 (i.e., stress path B^3D in Figure 3). According to the data by Jennings and Burland (1962) and more recently by Cunningham et al. (2003), the isotropic compression line in the space of void ratio against logarithmic mean stress will be curved, in a pattern similar to that shown in Figure 2 for $s > 0$. However, the isotropic compression path (B^3D) is clearly elastoplastic, not purely elastic. The preconsolidation (yield) stress correspondingly increases from zero to the current stress level. Similarly, a loading path (C^1C^3) followed by a drying path C^3D is also elastoplastic.

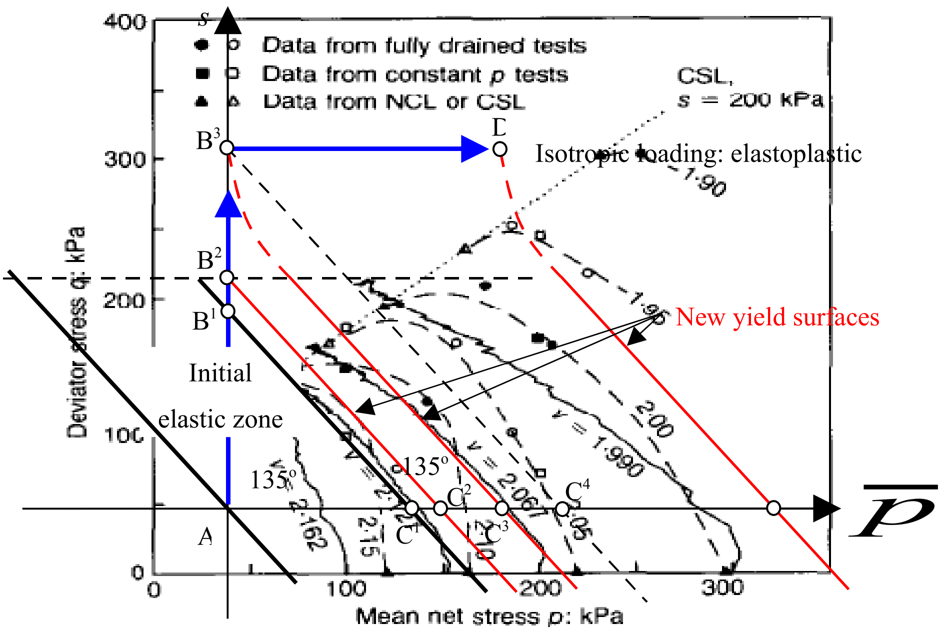

Figure 3. Expansion of the elastic zone during drying and compression of a slurry soil.

Figure 4. Specific volume contours at suction of 200 kPa (Wheeler and Sivakumar, 1995).

We also have difficulty to accept the general statement that the uniqueness of the state boundary surface is well accepted in the literature. The two references mentioned in the discussion both used data for compacted soils. The state boundary surface shown in Figure 14 in Delage and Graham (1996) is for the specific stress paths: isotropic compression under constant suction followed by suction reduction under constant stress. For such a stress path, the SFG model predicts similar volume changes to those presented in Figure 3a, even though the term ‘state boundary surface’ was not used in the SFG model. The so-called experimental evidence for the uniqueness of the state boundary surface refers to Figure 17 in Wheeler and Sivakumar (1995) and is cited above as Figure 4. It seems to us that this figure can to the best verify the uniqueness of the critical state line. The specific volume contours in the figure were obtained from triaxial shear tests under different stress paths but a fixed suction value (200kPa). It seems to us that the relationship between suction, net mean stress and specific volume is not necessarily unique.

In summary of this question, we do not think that the state boundary surface for a soil is necessarily unique. The stress-path dependency seems to be a good feature to have in a constitutive model for unsaturated soils, particularly when modelling unsaturated soils that are air-dried from slurry states.

Question 3: Modelling of Hysteresis of SWCC

This question is relevant and insightful. We agree that the hysteretic SWCC curves should depend on the void ratio of the soil and consequently on stress state. The hydraulic model used in the SFG model follows that in Sheng *et al.* (2004) and represents a simplification of real soil behaviour.

When an unsaturated soil specimen is sheared under undrained conditions (i.e. with constant water content), the degree of saturation can increase a little due to the volume contraction of the soil. This has been observed in laboratory tests (e.g. Sun *et al.* 2008). However, the recoverable portion of the volume change is generally small, if at all. In addition, the initial state is usually within the SI and SD surfaces, not on the SD surface. To reach the SD surface the soil must be highly wetted. Both the main drying and main wetting curves refer

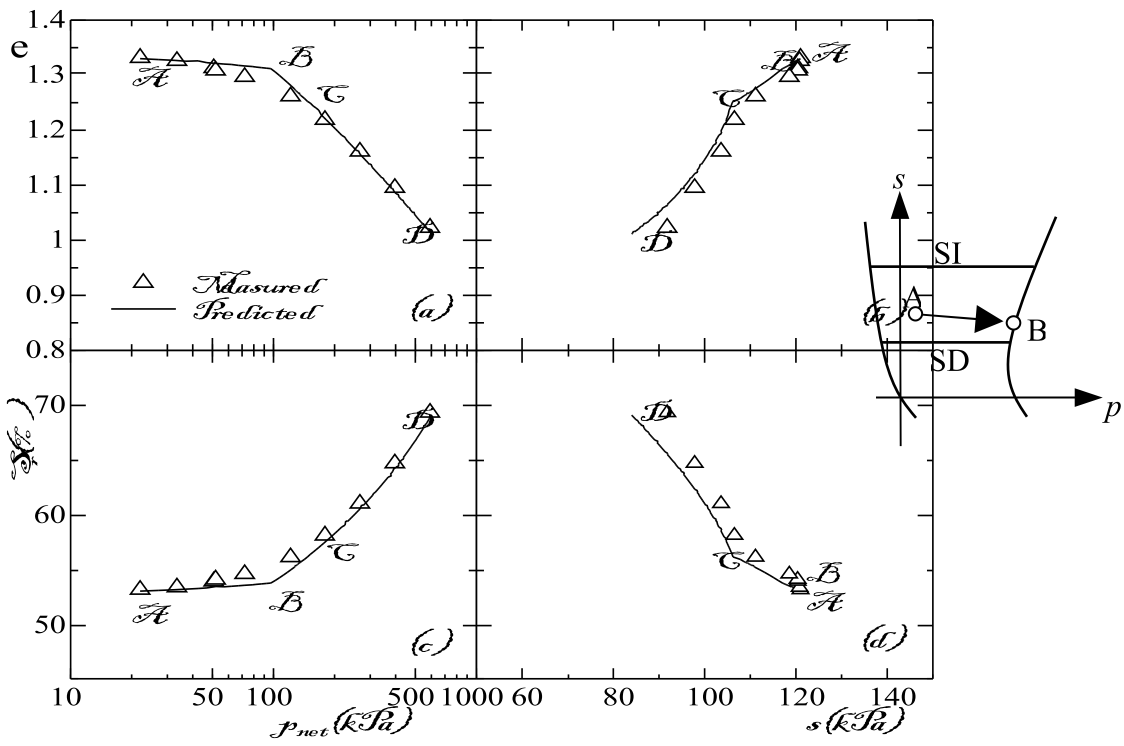

Figure 5. Isotropic compression test on compacted Pearl clay under undrained condition (Sun *et al.* 2008).

to drained conditions. Therefore, the elastic path (if any) AB is likely to be inside the SI and SD surfaces. In such a case, both the volume change and the change of degree of saturation are recoverable. We note that, once the volume change becomes irrecoverable, the change of degree of saturation will also be irrecoverable.

The question raised by the discussers may also apply to the compaction process (or isotropic compression) of the soil under constant water content conditions. Again, the SFG model can only model an elastic path AB that is within the SI and SD surfaces, if the volume change is recoverable. Figure 5 shows typical responses of Pearl clay during undrained isotropic compression. It is clear that the changes in suction and degree of saturation are quite small when the mean stress changes from A to B and causes only elastic volume change.

While the discussers may have amplified some effects that we feel are secondary, we note the issue raised by the discussers indeed constitutes a theoretical challenge to the SFG model. We also note that the case illustrated in the discussion cannot easily be modeled even if we incorporate a density-dependent SWCC curves, because the SWCC curves would usually shift to higher suctions with decreasing void ratios. The issue definitely warrants further research.

Conclusion

In summary, we do not think that mathematical constraints should be imposed on the parameters κ_{vs} and κ_{vp} or on λ_{vs} and λ_{vp} to make the volumetric model stress-path independent. Rather, these parameters should be based on the basic features of soil behaviour.

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