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Elasto-plastic Modelling of Unsaturated Soils : an Overview

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ABSTRACT: This paper presents an overview of elasto-plastic modelling of unsaturated soils. Alternative stress and strain variables that have been used in constitutive models for unsaturated soils are discussed. The use of an effective stress equation may simplify constitutive relations, but often result in a stress space that depends on the material state. In this case, the constitutive behaviour is embodied both in the stress definition and in the constitutive relation. Alternative volumetric models in the literature are then discussed. The focus is placed on the comparison of a new model with other existing models. This new model is formulated in terms of independent stress variables and features a smooth transition between saturated and unsaturated states. The implications of the volumetric model for yield stress and shear strength functions are discussed. The paper also shows how hysteresis in soil-water characteristic curves can be incorporated into the elasto-plastic framework, leading to a coupled hydro-mechanical model. Finally, the paper demonstrates the derivation of the incremental stress-strain relations for unsaturated soils and discusses briefly the implementation of these relations into the finite element method.

1 Introduction

Unsaturated soil mechanics did not emerge in parallel with saturated soil mechanics. There was a delay of about three decades before the basic principles of unsaturated soils began to be understood. Initial studies of unsaturated soils focused on the search for an effective stress equation for unsaturated soils. Then researchers focused on studies related to the volume change behaviour of expansive soils through a series of international conferences from 1965 to 1992. There was also limited consideration of collapsible soil behaviour during this period. It became obvious at each subsequent international conference that there was a desire to expand the scope of interest to include shear strength, contaminant transport, heat flow and other unsaturated soil phenomena. The subject of shear strength behaviour has proven to be easier to understand than volume change behaviour. The theories that emerged for volume change and shear strength took the form of extensions of classic forms that had been developed for saturated soils. With time it was obvious that the classic soil mechanics framework needed to be expanded to more adequately embrace elasto-plastic formulations for unsaturated soils.

In 1995 the First International Conference on Unsaturated Soils (Paris, France) was held. This conference witnessed an enlargement of the scope of unsaturated soils problems of interest to geotechnical engineers. Subsequent international conferences on unsaturated soils have been held in Beijing, China, Recife, Brazil and Carefree, Arizona. In addition, there have been a series of regional conferences on unsaturated soils held in Brazil, Southeast Asia, Europe and United States. There has been continuously increasing interest in understanding the fundamental behaviour of unsaturated soils, particularly in the shear strength, volume change and saturation change behaviour associated with suction changes. Significant progress has been made on understanding shear strength, volumetric and soil-water characteristic behaviour through 1970s to 1990s. On the other hand, constitutive models that are based on elasto-plasticity theory and accommodate a more complete stress-strain relationship started to emerge since early 1990s, and by now numerous such models exist in the literature. Unfortunately, the two approaches (i.e. the fundamental strength-volume-saturation-suction modelling and the elasto-plastic stress-strain modelling) have been advancing in an independent manner rather than embracing each other. In this paper, an different approach is attempted: an overview of elasto-plastic modelling of unsaturated soils is embodied in the discussion of the key fundamental issues: (1) the volumetric behaviour related to suction or saturation changes, (2) the strength behaviour related to suction or saturation changes, (3) the soil-water characteristic curves, and (4) the implementation of unsaturated soil models. In so doing, the advantages and disadvantages of alternative models are also discussed. This review is therefore somewhat different from previous reviews such as those by Gens (1995), Jommi (2000) and Gens *et al.* (2006).

2 Stress and strain variables

Constitutive relations used to represent mechanical behaviour of materials are usually described in stress space. The choice of the stress space is thus a fundamental issue in constitutive modelling. Idealistically the definition of stresses should be independent of the behaviour or the states of the material, so that the stress space does not change with the material state. There is little argument that total stresses should be used for describing the stress space of single phase materials such as metals and dry sands. It is also generally accepted by the soil mechanics community that effective stresses (i.e., difference between the total stresses and pore pressure) can be used to describe the mechanical behaviour of saturated soils. The definitions of the total stresses for dry soils and the effective stresses for saturated soils are independent of the soil behaviour or soil state. The stress spaces in these cases are thus separated from the material state. Also, the so-called total stress in dry sand is actually the difference between the total stress (i.e., force per unit cross-section area) and the atmospheric pore-air pressure.

For soils that are unsaturated with more than one pore fluid, the choice of the stress space appears to historically have become more complicated in that the stress space is used in a manner where it is dependent on the material state. Biot (1941) was probably the first to suggest the need to use two independent stress state variables for an unsaturated soil. In 1950s and 1960s, however, great efforts were made to identify a single effective stress that can be used to describe the deformation and strength characteristics of unsaturated soils, as the one used for saturated soils (e.g. Aitchson and Donald, 1956; Bishop, 1959). The following effective stress equation was proposed by Bishop (1959)

$$\sigma'_{ij} = \sigma_{ij} - u_{\mathrm{a}}\delta_{ij} + \chi\left(u_{\mathrm{a}} - u_{\mathrm{w}}\right) = \bar{\sigma}_{ij} + \chi s \tag{1}$$

where σ_{ij} is the total stress, σ'_{ij} is the Bishop effective stress, $\bar{\sigma}_{ij}$ is the net stress, u_{a} is the pore air pressure, u_{a} is the pore water pressure, *s* is the soil suction, χ is a parameter that may depend on the degree of saturation or on the soil suction. The soil suction in this paper refers to the matric or matrix suction which consists of the capillary and adsorptive potentials. When the water phase in soil pores is continuous, the capillary potential (ψ_{c}) is dominant in the matric suction $s \square \psi_{\mathrm{c}} = u_{\mathrm{a}} - u_{\mathrm{w}}$. When the pore water exists as adsorbed water films in the soil, the adsorptive potential (ψ_{a}) is dominant in the matric suction and the true water pressure is not well defined. In this case an apparent water pressure ($\bar{u}_{\mathrm{w}}$) can be used to quantify the adsorptive potential $s \square \psi_{\mathrm{a}} = u_{\mathrm{a}} - \bar{u}_{\mathrm{w}}$, i.e. the apparent water pressure represents the negative adsorptive potential measured in excess of air pressure. With such a definition, the matric suction can be used for a relatively large range of saturation, from fully saturated to very dry states.

Even though the new definition of effective stress in the 1960s has led to some success in describing the shear strength of unsaturated soils, it has not led to great success in modelling the general mechanical behaviour of unsaturated soils, not at least up to the late 1990s and early 2000s. Some limitations were early reported in using a singe Bishop effective stress in explaining volume collapse during wetting of unsaturated soils (Jennings and Burland, 1962). More importantly, because the parameter χ usually depends on material properties and even material states, the stress space defined by equation (4) usually depends on the material behaviour and changes with material states. Therefore, the constitutive behaviour of the material is embodied in both the constitutive relationships and the stress space where the constitutive relations are established.

In 1960s and 1970s, it was realised that it was possible to use two independent sets of stress variables to model unsaturated soil behaviour rather than combining them into one single effective stress as in the Bishop effective stress. For example, Coleman (1962) suggested the use of the net axial and radial stresses and the net pore water pressure to represent triaxial stress states. Bishop and Blight (1963) used the concepts of independent stress state variables when plotting volume changes in an unsaturated soil. Matyas and Radhakrisna (1968) used the independent stress variables (called 'state parameters') to describe the volumetric behaviour of unsaturated soils. Volume change was presented as 3D surfaces with respect to the independent stresses. Numerous other researchers have subsequently presented the volume change behaviour as surfaces defined by independent stress state variables (Aitchison and Martin, 1973; Alonso and Lloret, 1982).

Fredlund and Morgenstern (1976) presented volume-mass constitutive relations for a compacted kaolin, unsaturated soil. The results confirmed that two independent constitutive surface were required when mapping changes in volume-mass soil properties for an unsaturated soil. Fredlund and Morgenstern (1977) further provided a theoretical basis and justification for the use of two independent stress state variables. The justification was based on the superposition of coincident equilibrium stress fields for each of the phases of a multiphase system, within the context of continuum mechanics. Three possible combinations of independent stress state variables were shown to be justifiable from the theoretical continuum mechanics analysis. However, it was the net stress and the matric suction combination that proved to be the easiest to apply in engineering practice:

$$\begin{pmatrix} \sigma_{ij} - u_a \delta_{ij} \\ u_a - u_w \end{pmatrix} = \begin{pmatrix} \bar{\sigma}_{ij} \\ s \end{pmatrix} \tag{2}$$

The net normal stress primarily embraces the activities of humans which are dominated by applying and removing total stress (i.e., excavations, fills and applied loads). The matric suction stress state variable primarily embraces the impact of the climatic environment above the ground surface. Fredlund *et al* (1978) also presented a shear strength equation using the independent stress variables.

The liberation of the suction from a single effective stress has played a very positive role for the knowledge explosion in unsaturated soil mechanics in 1990s and 2000s. A number of events have witnessed the knowledge explosion since 1990s: the establishment of the International Conferences on Unsaturated Soils, the development of new constitutive models and testing techniques for unsaturated soils, and the publication of text books such as *Soil Mechanics for Unsaturated Soils* (Fredlund and Rahardjo, 1992). In the context of constitutive modelling, Alonso *et al*. (1990) for the first time provided a complete elastoplastic framework for modelling unsaturated soil behaviour. The model of Alonso *et al.* (1990) used the net stress and suction as the stress variables. This model was later referred to as the Barcelona Basic Model. A number of other elastoplastic models were developed in 1990s (e.g., Kohgo *et al*., 1993; Modaressi and Abou-Bekr, 1994; Wheeler and Sivakumar, 1995; Cui and Delage, 1996; Bolzon *et al*., 1996; Loret & Khalili, 2002, Tang and Graham, 2002; Oka *et al.*, 2006). All these models deal with stress-strain relations only. More recent models have incorporated suction-saturation relationships with hysteresis into stress-strain relationships (Vaunat *et al*., 2000; Wheeler *et al*., 2003; Gallipoli *et al*., 2003; Sheng *et al.*, 2004; Tamagnini, 2004; Sun *et al*., 2006; Santagiuliana and Schrefler, 2006).

A common feature of these models is that the suction is considered as an additional stress variable, or at least as an additional hardening parameter (Loret & Khalili, 2002). However, there is little consensus on whether an independent stress (e.g. net stress or total stress) or an effective stress variable should be used. A different argument was put forth by Houlsby (1997) using the work-conjugate variables in the work input for a soil element. In terms of work input, one difference in using a different set of stress variables is the change of their work-conjugate strain variables. For example, the work-conjugate strain variables to the two sets of independent stress variables (the net stresses and the suction), are the soil skeleton strains and the volumetric water content:

$$\begin{pmatrix} \sigma_{ij} - u_a \delta_{ij} \\ u_a - u_w \end{pmatrix} \Leftrightarrow \begin{pmatrix} \varepsilon_{ij} \\ \theta \end{pmatrix} \tag{3}$$

where ε_{ij} is the soil skeleton strain and θ is the volumetric water content. It is noted that the definitions of the stress variables in equation (3) are the same as those proposed by Fredlund and Morgenstern (1978) and are independent of each other and independent of material states. Their work-conjugate strain variables are not independent (i.e. the soil skeleton strain and the volumetric water content are dependent upon each other). It is also noted that the net stress becomes the total stress when the soil is saturated and the pore air is under atmospheric air pressure. When the air pressure is constantly at atmospheric pressure (which is approximately true for most insitu conditions), the matric suction is nothing else but negative pore water pressure. Early models using the net stress and suction thus suffer a major drawback, i.e. the continuity at the transition between saturated states and unsaturated states, because the stress variables used for the saturated soil behaviour is the Terzaghi effective stress.

On the other hand, the work-conjugate stress variables related to the two sets of independent strain variables (i.e., the soil skeleton strains ε_{ij} and the degree of saturation S_r) are the average stresses and a (modified) suction (*ns*). These strain variables are considered to be independent because a change in one of them does not necessarily result in a change in the other. The porosity *n* plays a role to scale the work input (due to a change in saturation) per unit void volume to that per unit volume of the soil matrix. Therefore, the second alternative stress variables take the following form (Sheng *et al*. 2004):

$$\begin{pmatrix} \sigma_{ij} - u_a \delta_{ij} + S_r (u_a - u_w) \\ (u_a - u_w) \end{pmatrix} \Leftrightarrow \begin{pmatrix} \varepsilon_{ij} \\ S_r \end{pmatrix} \tag{4}$$

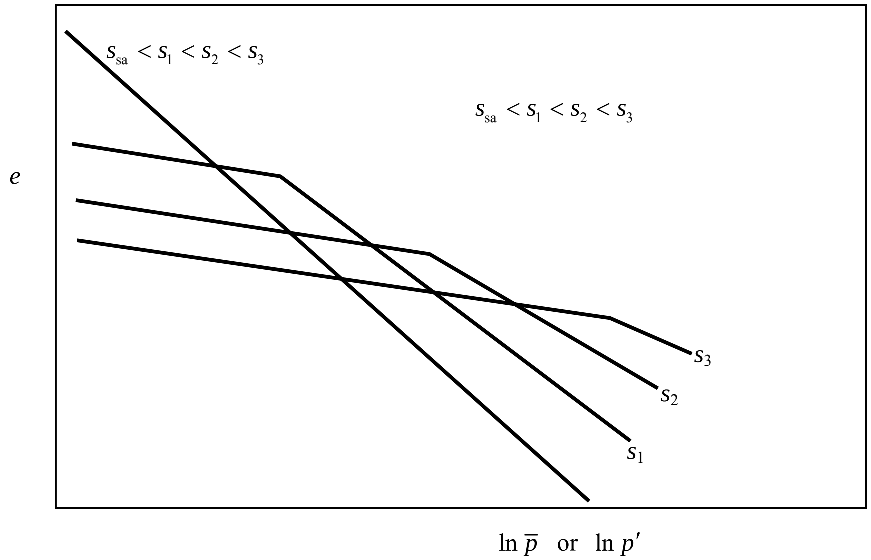


Figure 1. Qualitative prediction of void ratio versus mean stress under constant suctions according to Equation (4) or (5).

where S_r is the degree of saturation and n is the porosity. The stress variables defined in equation (6) are dependent on one another and as well as on the material state (S_r). However, the work-conjugate strain variables are independent of one another. The average stress becomes the Terzaghi effective stress when the soil becomes saturated. Therefore, the above stress variables can be used consistently for both saturated and unsaturated states. Consequently these stress variables have a significant advantage in terms of implementation into finite element codes. Such a set of stress variables have also been used in more recent models such as those by Sun *et al.* (2006) and Santagiuliana and Schrefler (2006). More recently Nuth and Laloui (2008) and Laloui and Nuth (2008) refer to this set of stresses as the generalised stresses for unsaturated soils and have provided further experimental evidence to endorse its use.

The more complex stress variables defined by equation (4) tend to lead to simpler constitutive equations, whereas the simpler stress variables defined by equation (3) tend to lead to more complex constitutive equations. The complex stress variables in equation (4) depend on material states and are not controllable in laboratory testing. Therefore, strictly speaking, it is not possible to develop a completely new constitutive relationship in terms of these variables, unless an existing framework is used. However, it is possible to transform an existing constitutive relationship formed in terms of the simpler stress variables (equation (3)) to the complex stress space. Such a transformation can often overcome the discontinuity problem at the transition between saturated and unsaturated states, as was done by Sheng *et al*. (2003a, 2003b) for the Barcelona Basic Model. The choice of the stress variables also has a significant influence on the yield and failure surfaces, which is later discussed in this paper.

3 Volumetric stress-strain models

Soil suction affects the volumetric behaviour, yield stress and shear strength of an unsaturated soil. Suction generates interparticle forces normal to particle contacts, while pore pressures generate isotropic stresses around soil particles. As such, soil suction plays a more complex role than the pressure or mean stress. As pointed out by Li (2003), some measure of soil fabric should be taken into consideration to accommodate the effects of suction. However, measures of soil fabric are difficult to achieve and have not yet been used in constitutive modelling. Instead, suction is usually treated as a similar quantity to the mean stress. Under such a framework, the only extra constitutive law that is absolutely required to extend a saturated soil model to unsaturated soils is the volume – stress – suction relationship. The effects of suction on the yield stress and shear strength can be incorporated into the model based on the volume – stress – suction relationship.

The change in the specific volume (v) of an unsaturated soil in response to suction (s) or mean stress (p) change is typically modelled in one of the following ways:

$$\mathrm{d}v = \frac{\mathrm{d}N_s}{\mathrm{d}s}\mathrm{d}s - \lambda_{vp}\frac{\mathrm{d}\overline{p}}{\overline{p}} - \ln\overline{p}\frac{\mathrm{d}\lambda_{vp}}{\mathrm{d}s}\mathrm{d}s \qquad \text{(Net stress, Alonso } et\ al.\text{, 1990)} \qquad (5)$$

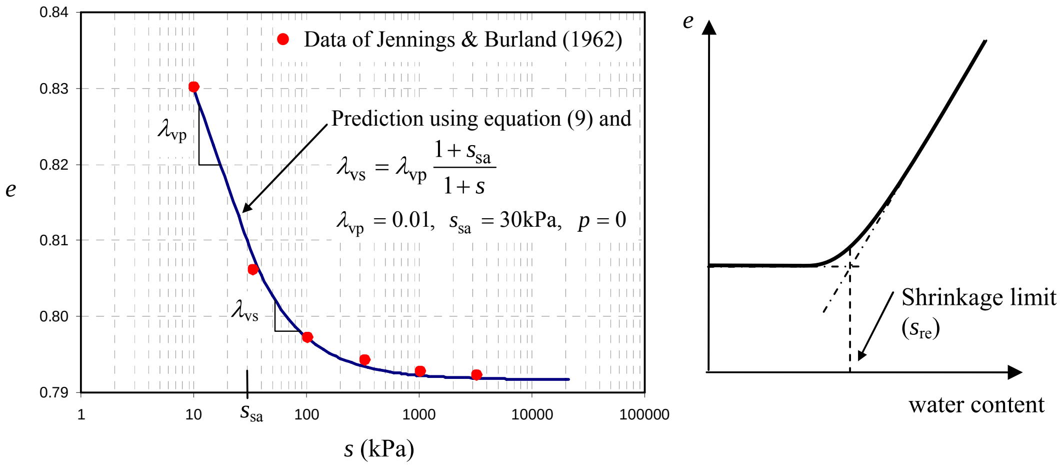


Figure 2. Schematic view of void ratio versus suction under zero net mean stress (s_{sa}: suction corresponding to full saturation, s_{re}: residual suction).

$$\mathrm{d}v = \frac{\mathrm{d}N_s}{\mathrm{d}s}\mathrm{d}s - \lambda_{vp}\frac{\mathrm{d}p'}{p'} - \ln p'\frac{\mathrm{d}\lambda_{vp}}{\mathrm{d}s}\mathrm{d}s \qquad \text{(Effective stress)} \qquad (6)$$

$$\mathrm{d}v = -\lambda_{vp}\frac{\mathrm{d}\bar{p}}{\bar{p}} - \lambda_{vs}\frac{\mathrm{d}s}{s} \qquad \text{(Net stress, Fredlund \& Rahardjo, 1993)} \qquad (7)$$

where N_s is the specific volume of the soil when the mean stress is unity, λ_{vp} is a material parameter representing the stress compressibility under constant suction, $\bar{p}$ is the net mean stress, p' is the effective mean stress, λ_{vs} is a material parameter representing the suction compressibility under constant mean stress. The parameter λ_{vp} in equations (5) and (6) and the parameter λ_{vs} in equation (7) are usually considered as functions of suction. Over a certain stress range, the parameter λ_{vp} is usually approximated by one or two constants, depending on the preconsolidation pressure.

Equation (5) uses the net stress $\bar{p}$ and is used in e.g. the Barcelona Basic Model of Alonso *et al* (1990) and also in many other models. Equation (6) uses the effective stress and is used by Kohgo *et al.* (1994) and Loret and Khalili (2002). Equation (7) uses the net stress $\bar{p}$ and separates the compressibility due to a stress change (λ_{vp}) from that due to a suction change (λ_{vs}).

All these equations are of course based on the Cam Clay elastoplasticity for saturated soils:

$$\mathrm{d}v = -\lambda_{vp}\frac{\mathrm{d}p'}{p'} = -\lambda_{vp}\frac{\mathrm{d}p}{p+(-u_w)} - \lambda_{vp}\frac{\mathrm{d}(-u_w)}{p+(-u_w)} \qquad \text{(Saturated soils)} \qquad (8)$$

Equations (5) to (8) are confusingly similar, but they bear different implications. For example, equations (5) and (7) do not recover equation (8) for saturated states. A simple verification for this is to consider the case where the pore air pressure (u_a) remains atmospheric. Under this condition, equations (5) and (7) are only valid for zero pore water pressure when the soil becomes saturated. The atmospheric air pressure can be treated as zero pore air pressure and suction as a negative pore water pressure for all saturated states. When the pore air pressure remains constantly at the atmospheric pressure, the net stress becomes the total stress and suction becomes the negative pore water pressure. Such a concept will provide a continuous transition between saturated and unsaturated states.

Equation (6) fully recovers the model for saturated soils, but contains the effective stress parameter χ. This parameter often depends on the material as well as the material state, leading to the questionable outcome that the stress space where the material behaviour is modelled changes with the material behaviour and even the material state. Equation (7) separates the compressibility due to a stress change (λ_{vp}) from that due to a suction

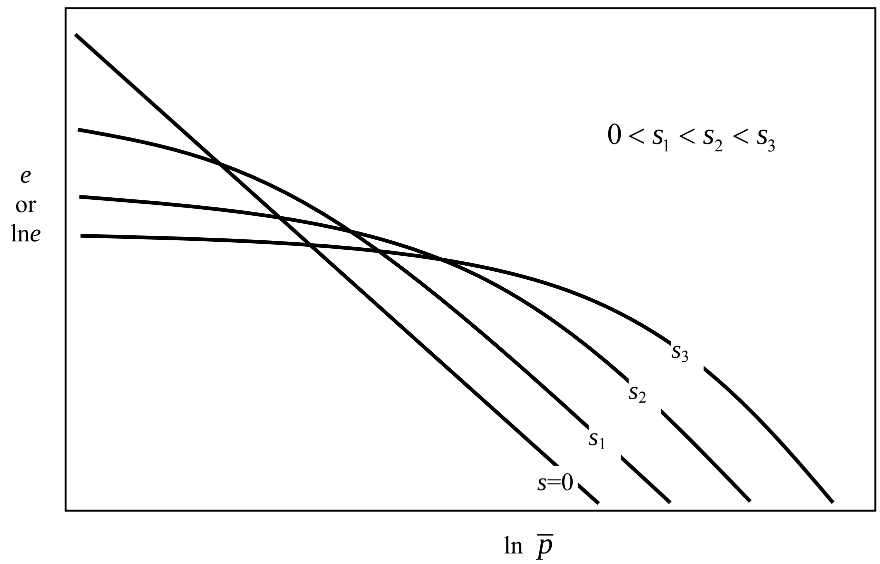


Figure 3. Qualitative prediction of normal compression lines under constant suctions according to Equation (9) or (11).

change (λ_{vs}). As such, the parameter λ_{vp} is not necessarily a function of suction, which is an advantage over the other two equations.

A schematic view of the predictions according to equations (5) and (6) is shown in Figure 1. These two models both have difficulties in explaining the curvature of the normal compression lines at positive suctions for soils dried from slurry. Let us consider the case where a slurry soil is first dried to a specific suction and then isotropically compressed at that suction. The isotropic compression line for this soil is usually curved in the *e* – ln*p* space, as shown by Jennings and Burland (1962). To approximate the curve using equation (5) or (6), we would have to use the overconsolidation concept, so that the curve is approximated by an initial elastic portion followed by an elastoplastic portion, as illustrated in Figure 1. However, the soil has never been over-consolidated, since the process of drying and loading does not involve any stress or suction decrease.

More recently, Sheng, Fredlund and Gens (2006) proposed the following model for the volumetric behaviour of unsaturated soils:

$$\mathrm{d}v = -\lambda_{vp}\frac{\mathrm{d}\overline{p}}{\overline{p}+s} - \lambda_{vs}(s)\frac{\mathrm{d}s}{\overline{p}+s} \qquad \text{(Net stress, Sheng } et\ al.\text{, 2006)} \qquad (9)$$

where the slope λ_{vp} can be independent of suction, and the slope λ_{vs} varies between λ_{vp} for saturated states and zero for suctions above the residual suction (Figure 2). This model

1. recovers the equation for saturated states, i.e. equation (8),
2. separates the compressibility due to stress and suction changes, and
3. can predict the smooth curvature of the normal compression lines under constant suctions for soils dried from slurry, without the use of the 'overconsolidation' concept (see Figure 3).

Equation (9) is very similar to equation (8), but with the negative pore water pressure replaced by the suction, and the total mean stress replaced by the net mean stress. It is also reasonable to state that a change in suction does not necessarily have the same effect as a change in mean stress once the soil becomes unsaturated. Sheng, Fredlund and Gens (2006) showed that equation (9) can capture a number of important features in unsaturated soil behaviour and can well represent experimental data.

Another issue related to all the above volumetric models pertains to the suction ranges and soil types where the models can be applied. First, all these models apply only to a continuum and become invalid once desiccation and cracking occurs. Second, for dry granular soils where the water phase becomes discontinuous, the concept of capillary suction may not fully apply. In this case, it is better to view suction as simply the affinity of the soil for water or the energy required to remove the water from the soil. More study is required on the behaviour of soils with water contents less than residual values.

A dry sand behaves in a similar manner to a saturated sand under fully drained condition. This phenomenon may not be able to be predicted using the concept of soil suction. In addition, the base model for saturated soils, (i.e., equation (8) is known to be more applicable to clayey soils than to granular soils). Nevertheless, other constitutive models used for saturated soils can be generalised to accommodate unsaturated soils in a similar manner. For example, Sheng *et al.* (2007) showed that the following equation well predicts the volume change of saturated or dry sands.

$$\frac{de}{e} = -\lambda_{\mathrm{vp}} \frac{dp'}{p' + p_{\mathrm{re}}} \qquad \text{(Saturated sands, Sheng } et\ al. \text{ 2007b)} \qquad (10)$$

where e is the void ratio, and p_{re} is a shifting stress depending on the initial void ratio of the soil as well as λ_{vp}. The shift stress can also be interpreted as the stress level where significant particle crushing occurs. Note that the parameter λ_{vp} in equation (10) is the slope in the double logarithmic $\ln e - \ln(p' + p_{\mathrm{re}})$ space. If equation (10) is used for a saturated sand, Equation (9) can be modified as follows to suit the unsaturated sand

$$\frac{\mathrm{d}e}{e} = \begin{cases} -\lambda_{\mathrm{vp}} \dfrac{\mathrm{d}\overline{p}}{\overline{p} + p_{\mathrm{re}} + s} - \lambda_{\mathrm{vs}}(s) \dfrac{\mathrm{d}s}{\overline{p} + p_{\mathrm{re}} + s} & s \le s_{\mathrm{re}} \\ -\lambda_{\mathrm{vp}} \dfrac{\mathrm{d}\overline{p}}{\overline{p} + p_{\mathrm{re}} + s_{\mathrm{re}}} & s > s_{\mathrm{re}} \end{cases} \qquad (11)$$

where s_{re} is the residual suction (see Figure 2). A threshold suction (s_{re}) is introduced in the equation (11) and above this value suction has no effect on the volume change. Because the residual suction for sands is relatively small (<100kPa) compared to particle crushing pressure of sands (1-100MPa), the effect of suction on volume change is relatively limited. Setting p_{re}=0 in Equation (10) recovers equation (8). Therefore, equation (11) can also be used for clays (with p_{re}=0).

4 Yield surface and hardening law

Soil suction is an additional stress variable and therefore it is necessary to determine the variation of the yield stress with suction, or the extension of the yield surface in the stress – suction space. The yield surface for an isotropic hardening soil usually represents the contours of the plastic volumetric strain (i.e. the hardening parameter). As such, the variation of the yield stress with suction can be derived from the volumetric model. For example, for the volumetric model defined by equation (5), it is possible to show that the following function represents the contours of plastic volumetric strain in the $\overline{p} - s$ space (see Sheng, Fredlund and Gens, 2006):

$$\overline{p}_{\mathrm{c}} = \begin{cases} \overline{p}_{\mathrm{c0}} - s & s \le s_{\mathrm{sa}} \\ \overline{p}_{\mathrm{r}} \left(\dfrac{\overline{p}_{\mathrm{c0}} - s_{\mathrm{sa}}}{\overline{p}_{\mathrm{r}}} \right)^{\frac{\lambda_{\mathrm{vp0}} - \kappa}{\lambda_{\mathrm{vp}} - \kappa}} & s > s_{\mathrm{sa}} \end{cases} \qquad (12)$$

where $\overline{p}_{\mathrm{c}}$ is the yield stress at suction s, $\overline{p}_{\mathrm{c0}}$ is the yield stress at suction s_{sa}, $\overline{p}_{\mathrm{r}}$ is a reference mean stress, λ_{vp0} is the slope of the normal compression line for saturated states, λ_{vp} is the slope of the normal compression line for unsaturated states (suction s), and κ is the slope of the unloading-reloading line for saturated states. Equation (12) defines the so-called loading-collapse (LC) yield surface in the Barcelona Basic Model (BBM) by Alonso *et al.* (1990).

A schematic view of the loading collapse yield surface defined by equation (12) is shown in Figure 4a. A number of observations can be made here. First, the yield surface is usually shown in the literature for suctions above the saturation suction only. Since the net stress becomes the total stress for saturated states, the yield surface actually follows the 45$^{\mathrm{o}}$ line for $s<s_{\mathrm{sa}}$. Second, the yield stress $\overline{p}_{\mathrm{c}}$ increases with increasing suction only if (1) $\lambda_{\mathrm{vp}} < \lambda_{\mathrm{vp0}}$ and $\overline{p} > \overline{p}_{\mathrm{r}}$, or (2) $\lambda_{\mathrm{vp}} > \lambda_{\mathrm{vp0}}$ and $\overline{p} < \overline{p}_{\mathrm{r}}$. These two alternative conditions are prerequisite to modelling the wetting-induced collapse. Third, an additional yield stress, $\overline{p}_0$, representing the apparent tensile strength for s> 0 must be defined (see Figure 4a). The following apparent tensile strength is used in the BBM:

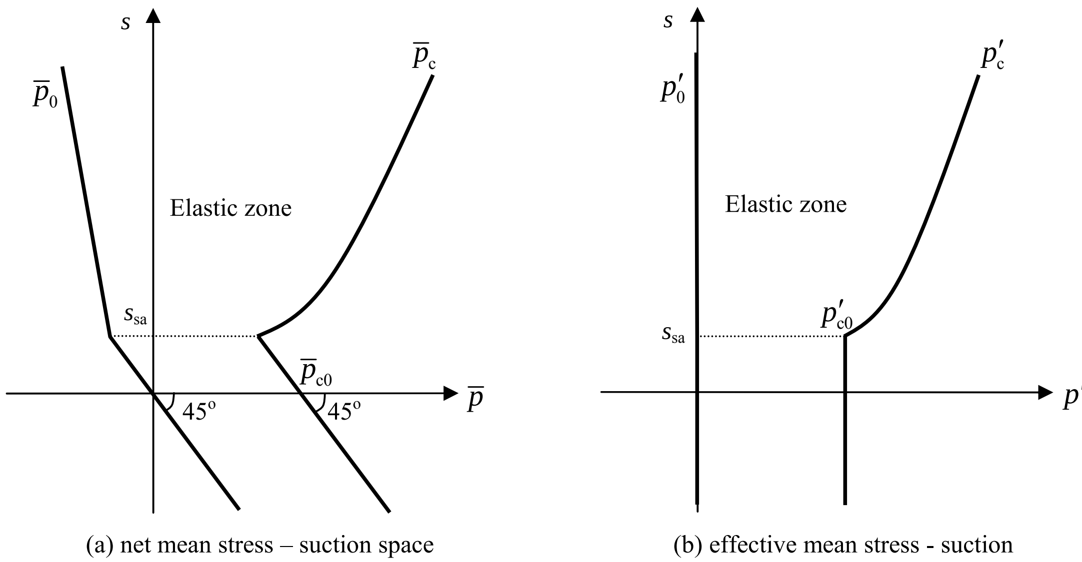


Figure 4. Schematic view of loading-collapse yield surface in mean stress – suction space.

$$\overline{p}_0 = \begin{cases} -s & s < s_{sa} \\ -\alpha s & s \geq s_{sa} \end{cases} \tag{13}$$

On the other hand, if the volumetric model is based on the effective stress (i.e. equation (6) is used), the corresponding yield stress becomes:

$$p'_c = \begin{cases} p'_{c0} & s \leq s_{sa} \\ p'_r \left(\dfrac{p'_{c0}}{p'_r} \right)^{\frac{\lambda_{vp0} - \kappa}{\lambda_{vp} - \kappa}} & s > s_{sa} \end{cases} \tag{14}$$

$$p'_0 = 0 \tag{15}$$

In equation (14), p'_r is a reference mean stress and $p'_r = 1$ if the soil specific volume at p'_r is given by parameter $N(s)$, see equation (6). The yield surface p'_c is shown schematically in Figure 4b. Because of the use of Bishop's effective stress, it is usually assumed that an apparent cohesion is zero. In addition, the loading-collapse yield surface extends to the saturated zone following a vertical line. However, the same conditions ($\lambda_{vp} < \lambda_{vp0}$ and $p' > p'_r$, or $\lambda_{vp} < \lambda_{vp0}$ and $p' < p'_r$) apply to the simulation of wetting-induced collapse.

The loading-collapse yield surfaces given in Figure 4 can not be used for a soil dried from slurry. Drying a slurry soil is similar to compressing the soil. Therefore, the stress state should always be on the current yield surface, which is clearly not possible in Figure 4a. Instead, Figure 4a would predict a purely elastic response for drying a slurry soil under constant stress. In Figure 4b, the stress state could be on the current yield surface if the effective mean stress increases at a faster rate that the yield stress as suction increases, which would then lead to additional conditions regarding the definition of the effective stress concept.

The model of Sheng, Fredlund and Gens (2006) provides a smooth transition between saturated and unsaturated states. The yield stresses, $\overline{p}_0$ and $\overline{p}_c$, can be derived from the volumetric model, i.e. equation (9) or (11), provided an explicit function is given to the parameter λ_{vs} is given. For example, the following yield stresses can be derived from equation (9) for a slurry soil:

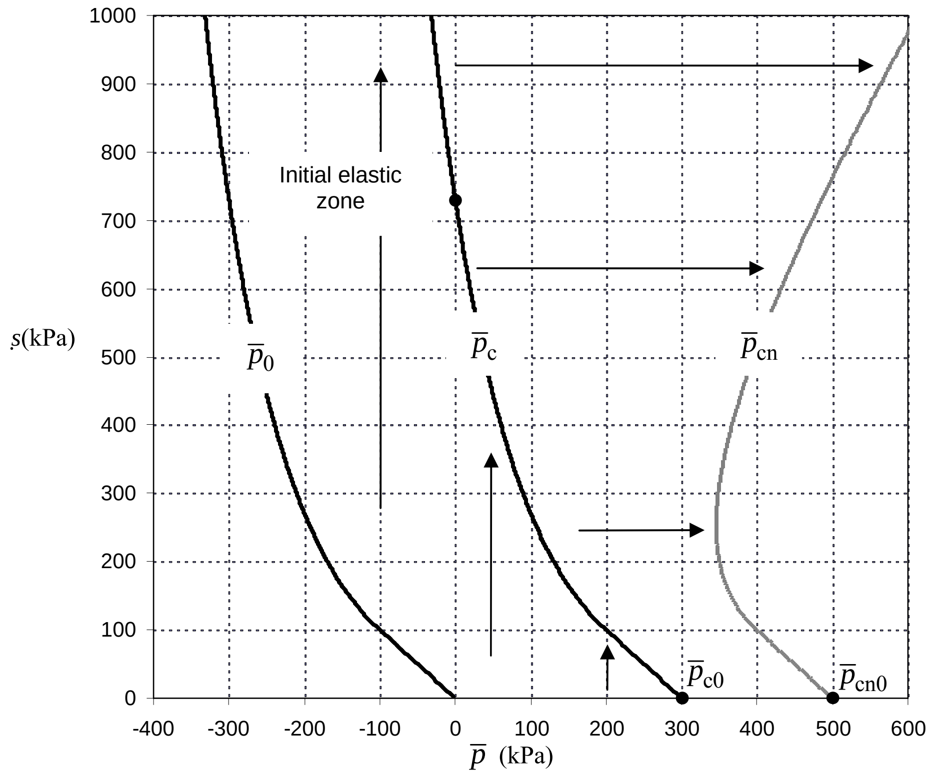


Figure 5. Initial yield surface for a soil that was consolidated to 300kPa at zero suction and its evolution when the soil is then compressed at different suction levels (s_{sa}=100kPa).

$$\overline{p}_c = \begin{cases} \overline{p}_{c0} - s & s < s_{sa} \\ \overline{p}_{c0} - s_{sa} - (s_{sa}+1)\ln\dfrac{s+1}{s_{sa}+1} & s \ge s_{sa} \end{cases} \tag{16}$$

This yield stress decreases with increasing suction. Therefore, drying a slurry soil will always be on the current yield surface. For a slurry soil, $\overline{p}_{c0}$ is zero. The apparent tensile strength caused by suction is then given as

$$\overline{p}_0 = \begin{cases} -s & s < s_{sa} \\ -s_{sa} - (s_{sa}+1)\ln\dfrac{s+1}{s_{sa}+1} & s \ge s_{sa} \end{cases} \tag{17}$$

However, for an unsaturated soil that is compressed or compacted at a suction above the saturation suction, the yield stress $\overline{p}_c$ in the model by Sheng, Fredlund and Gens (2006) changes to:

$$\overline{p}_{cn} = \begin{cases} \overline{p}_{cn0} - s & s < s_{sa} \\ \dfrac{\overline{p}_{cn0}}{\overline{p}_{c0}}\left(\overline{p}_{c0} + s - s_{sa} - (s_{sa}+1)\ln\dfrac{s+1}{s_{sa}+1}\right) - s & s \ge s_{sa} \end{cases} \tag{18}$$

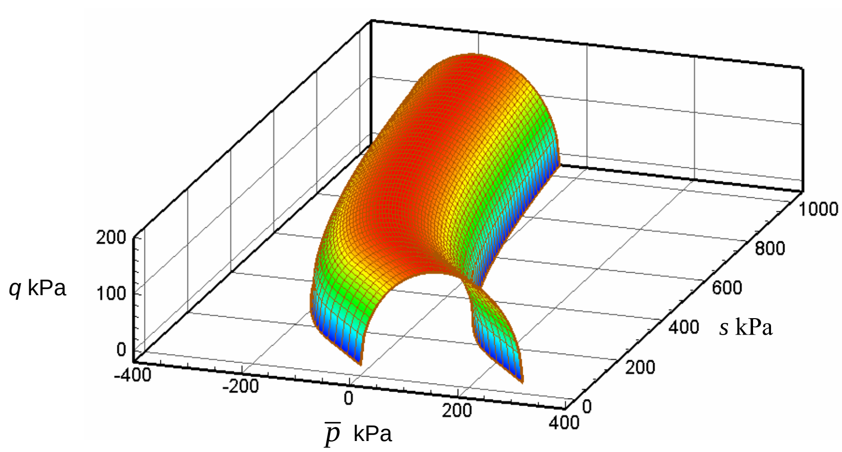
where $\overline{p}_{c0}$ is the initial preconsolidation pressure at zero suction (Not at s_{sa}), $\overline{p}_{cn0}$ is the new preconsolidation pressure at zero suction (see Figure 5).

The yield stresses given by equations (16)-(18) are illustrated in Figure 5. The curve $\overline{p}_0$ represents the apparent tensile strength of the soil caused by suction. The curve $\overline{p}_{\mathrm{c}}$ represents the yield stress if the soil is air-dried. For example, for a slurry soil that was first consolidated to 300 kPa and then air-dried at zero mean stress, the suction that causes plastic yielding is 730kPa. If the air-dried soil is compressed under constant suctions, the new yield stresses are then represented by the curve $\overline{p}_{\mathrm{cn}}$. Therefore, $\overline{p}_{\mathrm{c}}$ represents the yield stress for an air-dried slurry soil and $\overline{p}_{\mathrm{cn}}$ represents the yield stress for a compacted soil. The yield stress increases with increasing suction along the curve $\overline{p}_{\mathrm{cn}}$, not $\overline{p}_{\mathrm{c}}$. Therefore, wetting-induced collapse is only relevant for compacted soils with this model. It should also be noted that: (1) the transition between saturated and unsaturated states is continuous and smooth along all the three yield stresses; (2) like the models in Figure 4, the yield surfaces $\overline{p}_{\mathrm{c}}$ and $\overline{p}_{\mathrm{cn}}$ are non-convex in the $\overline{p}-s$ space; (3) the model is stress path dependent and different stress paths result in different yield surfaces.

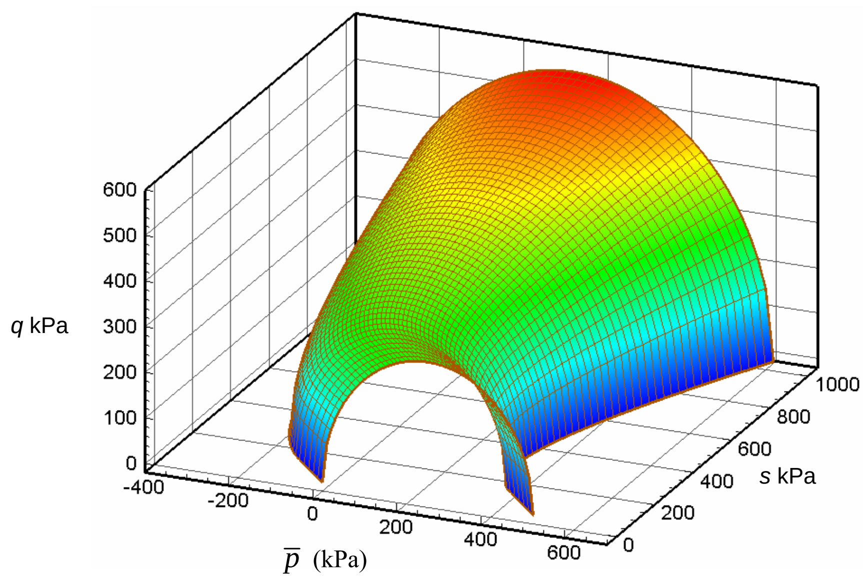
If the Modified Cam Clay model is used as the base model for the saturated soil, the elliptic yield surface can be extended to the suction axis according to equation (16)-(18):

$$f = q^2 - M^2(\overline{p} - \overline{p}_0)(\overline{p}_{\mathrm{y}} - \overline{p}) \equiv 0 \tag{19}$$

where *f* is the yield function, *q* is the deviator stress, *M* is a shear strength parameter and represents the slope of the critical state line in *q* – *p* space. The yield surface according to equation (19) is shown in Figure 6 for two types of unsaturated soils.



(a) Yield surface for air-dried soil from slurry.



(b) Yield surface for compacted soil.

Figure 6. Modified Cam Clay yield surfaces extended to suction axis (s_{sa}=100kPa).

In the three models mentioned above, the evolution of the yield surfaces is governed by the plastic volumetric strain. The hardening laws are written respectively as

$$\mathrm{d}\varepsilon_{\mathrm{v}}^{\mathrm{p}} = \frac{\lambda_{\mathrm{vp}} - \kappa}{v}\frac{\mathrm{d}\bar{p}}{\bar{p}} + \frac{\ln \bar{p}}{v}\frac{\mathrm{d}\lambda_{\mathrm{vp}}}{\mathrm{d}s}\mathrm{d}s \qquad \text{for equation (12)} \qquad (20)$$

$$\mathrm{d}\varepsilon_{\mathrm{v}}^{\mathrm{p}} = \frac{\lambda_{\mathrm{vp}} - \kappa}{v}\frac{\mathrm{d}p'}{p'} + \frac{\ln p'}{v}\frac{\mathrm{d}\lambda_{\mathrm{vp}}}{\mathrm{d}s}\mathrm{d}s \qquad \text{for equation (14)} \qquad (21)$$

$$\mathrm{d}\varepsilon_{\mathrm{v}}^{\mathrm{p}} = \frac{\lambda_{\mathrm{vp}} - \kappa_{\mathrm{vp}}}{\bar{p}+s}\mathrm{d}\bar{p} + \frac{\lambda_{\mathrm{vs}} - \kappa_{\mathrm{vs}}}{\bar{p}+s}\mathrm{d}s \qquad \text{for equations (16) and (18)} \qquad (22)$$

where $\varepsilon_{\mathrm{v}}^{\mathrm{p}}$ is the plastic volumetric strain, κ_{vp} is the elastic counterpart of λ_{vp} for unloading and reloading, and κ_{vs} is the elastic counterpart of λ_{vs}.

5 Shear strength with suction

The shear strength of an unsaturated soil is usually a function of suction. Fredlund *et al.* (1978) proposed the following relationship which conveniently separates the shear strength due to stress from that due to the matric suction:

$$\tau = \left[c' + \left(\sigma_{\mathrm{n}} - u_{\mathrm{a}}\right)\tan\phi'\right] + \left[\left(u_{\mathrm{a}} - u_{\mathrm{w}}\right)\tan\phi^{\mathrm{b}}\right] = \overline{c} + \left(\sigma_{\mathrm{n}} - u_{\mathrm{a}}\right)\tan\phi' \tag{23}$$

where τ is the shear strength, c' is the effective cohesion and is usually zero unless the soil is cemented, σ_{n} is the normal stress on the failure plan, ϕ' is the effective friction angle of the soil, ϕ^{b} is the frictional angle due to suction, and $\overline{c}$ is the apparent cohesion which includes the friction due to suction. If ϕ^{b} is set to ϕ' in equation (23), the effective stress principle for saturated soils is recovered.

The above shear strength equation was originally published in a linear form but it was later realized that experimental results showed the latter portion of the above equation to be nonlinear. The shear strength due to soil suction commences to deviate from the effective angle of internal friction at approximately the air entry value of the soil. The soil suction versus shear strength relationship then appears to have a gradual curvature until residual suction conditions are reached. Once residual suction conditions are reached the shear strength envelope remains approximately constant (or level at the same value) as suctions are increased. However, it is possible for the shear strength to decrease for sand soils and increase for clay soils as suctions are increased beyond residual conditions.

There are a number of models available in the literature for determining the friction angle ϕ^{b} (Vanapalli *et al.*, 1996; Fredlund *et al*, 1996; Oberg and Salfors, 1997; and Bao *et al*, 1998). In elasto-plastic models, the shear strength of an unsaturated soil is usually embodied in the apparent tensile strength function $\overline{p}_0$. For example, the apparent cohesion in the BBM is given as

$$\overline{c} = -\overline{p}_0 \tan\phi' = \begin{cases} s\tan\phi' & s < s_{\mathrm{sa}}, c' = 0 \\ \alpha s \tan\phi' & s \ge s_{\mathrm{sa}}, c' = 0 \end{cases} \tag{24}$$

The friction angle ϕ^{b} is then given by

$$\tan\phi^{\mathrm{b}} = \begin{cases} \tan\phi' & s < s_{\mathrm{sa}} \\ \alpha\tan\phi' & s \ge s_{\mathrm{sa}} \end{cases} \tag{25}$$

In this case, the friction angle ϕ^{b} is independent on suction.

In the model by Sheng, Fredlund and Gens (2006), the apparent cohesion due to suction is

$$\overline{c} = -\overline{p}_0 \tan\phi = \begin{cases} s\tan\phi & s < s_{\mathrm{sa}}, c' = 0 \\ \tan\phi\left(s_{\mathrm{sa}} + \left(s_{\mathrm{sa}} + 1\right)\ln\dfrac{s+1}{s_{\mathrm{sa}}+1}\right) & s \ge s_{\mathrm{sa}}, c' = 0 \end{cases} \tag{26}$$

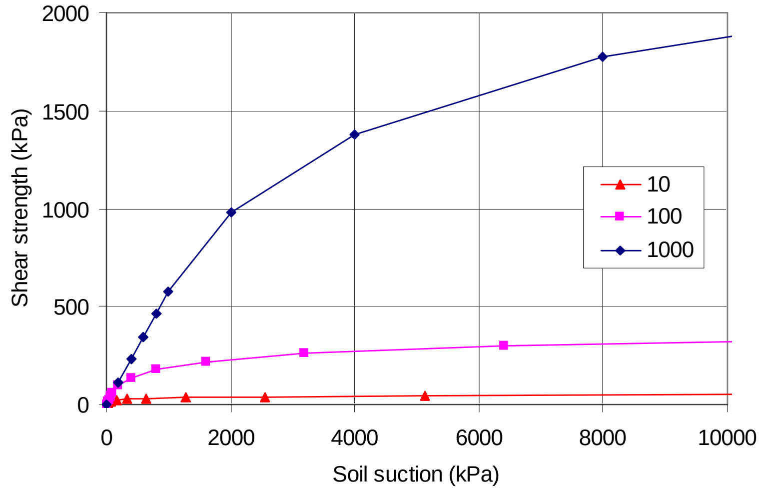


Figure 7 Shear strength versus soil suction up to 10,000 kPa for soils with air entry values of 10, 100, and 1000 kPa, predicted by equation (27).

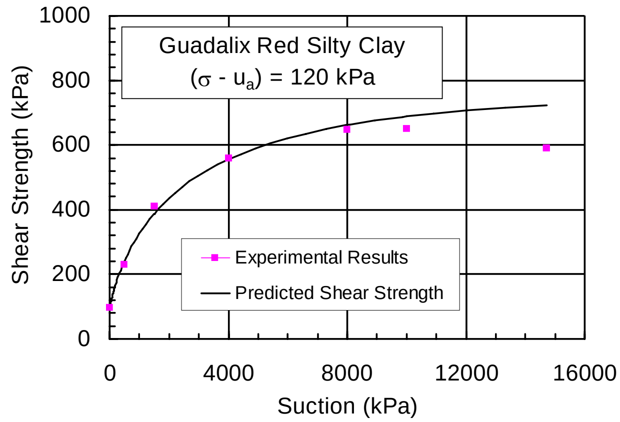


Figure 8. Predicted shear strength values for Guadalix Red silty Clay according to equation (27) (data taken from Escario and Jucá, 1989).

Therefore, the friction angle ϕ^{b} is given by:

$$\tan\phi^{\mathrm{b}} = \begin{cases} \tan\phi' & s < s_{\mathrm{sa}} \\ \tan\phi'\left(\dfrac{s_{\mathrm{sa}}}{s} + \left(\dfrac{s_{\mathrm{sa}}+1}{s}\right)\ln\dfrac{s+1}{s_{\mathrm{sa}}+1}\right) & s \geq s_{\mathrm{sa}} \end{cases} \quad (27)$$

In this case, the friction angle ϕ^{b} is a function of suction as well as the saturation suction. The predicted shear strength variation with suction is shown in Figure 7 and compared with experimental data for Guadalix Red silty Clay in Figure 8. It is shown that the prediction of equation (27) is very reasonable, at least qualitatively.

On the other hand, if Bishop's effective stress is used, the shear strength is usually assumed to be unique in the effective stress space:

$$\tau = c' + \sigma_{\mathrm{n}}' \tan\phi' = c' + \left(\sigma_{\mathrm{n}} - u_{\mathrm{a}}\right)\tan\phi' + \chi(u_{\mathrm{a}} - u_{\mathrm{w}})\tan\phi' \quad (28)$$

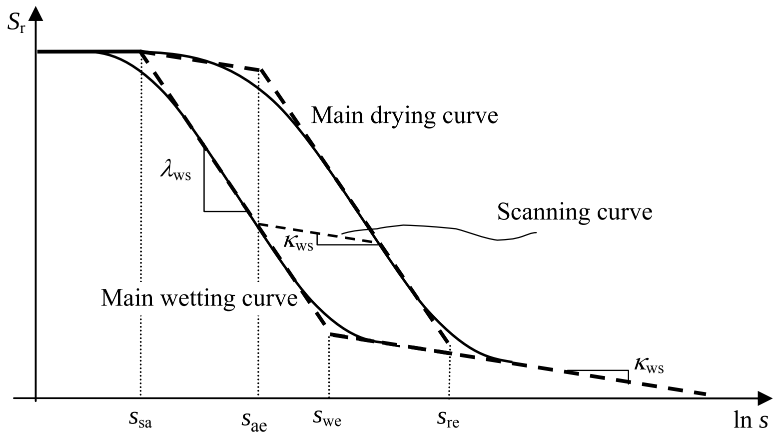


Figure 9. Degree of saturation versus suction (dashed lines represent simplification).

The above equation also implies that $\tan\phi^{b} = \chi \tan\phi'$. Recently Nuth and Laloui (2008) provided some experimental evidence for the uniqueness of c' and ϕ' in Bishop's effective stress space with $\chi = S_r$.

6 Hysteresis of soil-water characteristics

Extensive research has been done on the soil-water characteristic curves for a soil, first in the field of soil physics and later within geotechnical engineering (see, e.g., Hillel, 1982; Fredlund and Rahardjo, 1993). The soil-water characteristic curve (SWCC) is usually presented in space as the volumetric water content (θ) versus soil suction or in space as the degree of saturation (S_r) versus suction. A number of empirical θ – s relations exist in the literature, and the ones that are commonly used include that of Gardner (1958), van Guenuchten (1980) and Fredlund and Xing (1994). These relations are usually written as continuous functions and do not explicitly consider the hysteretic behaviour during a drying-wetting loop. However, in an elasto-plastic modelling framework which must predict the responses for all possible wetting and drying paths, an incremental form of between ds and dθ or between ds and dS_r is preferred. Recently Li (2005) presented an incremental soil-water characteristic relationship between ds and dS_r. This incremental SWCC model includes smooth hysteretic responses to arbitrary wetting/drying paths, and can be incorporated into elasto-plastic models for unsaturated soils. However, the model by Li (2005) follows the bounding surface framework which is different from the classic elasto-plasticity framework discussed in this paper. More recently, Pedroso and Sheng (2008) have developed an incremental saturation-suction relationship that incorporates the hysteretic behaviour. Their model is formulated in the same framework as elasto-plasticity and can be conveniently incorporated into an elasto-plastic stress – strain relation. The detail of the model of Pedroso and Sheng (2008) can be found in the proceedings of this conference. In this paper, a very simple model presented by Sheng, Fredlund and Gens (2006) is described. This simple model does not consider the hysteretic behaviour within the main drying and main wetting curves (see Figure 9).

As a simple approximation, a piece-wise linear relationship between the degree of saturation S_r and logarithmic soil suction can be assumed:

$$dS_r = -\lambda_{ws}\frac{ds}{s} \tag{29}$$

where the slope λ_{ws} may change with suction. For soil suctions below the saturation suction, the soil is saturated and the degree of saturation remains essentially constant. For soil suctions larger than the residual suction, the water content gradually decreases to zero at a suction of 10^6 kPa (Fredlund and Rahardjo, 1993). The slope λ_{ws} is assumed to be constant between the air entry and the residual suction for a drying soil (Wheeler *et al*., 2003). Therefore, we have for increasing suction, as shown in Figure 9:

$$\lambda_{ws} = \begin{cases} 0 & s < s_{sa} \\ \kappa_{ws} & s_{sa} \le s < s_{ae} \\ \lambda_{ws} & s_{ae} \le s < s_{re} \\ \kappa_{ws} & s \ge s_{re} \end{cases} \tag{30}$$

where s_{ae} is the air entry value, and s_{re} is the residual suction (see Figure 9). The above equation is only valid for the main drying curve. For the main wetting curve and the scanning curve, the slope must be adjusted accordingly (see Figure 9). The soil suction versus water content relationship is affected by the mean net stress

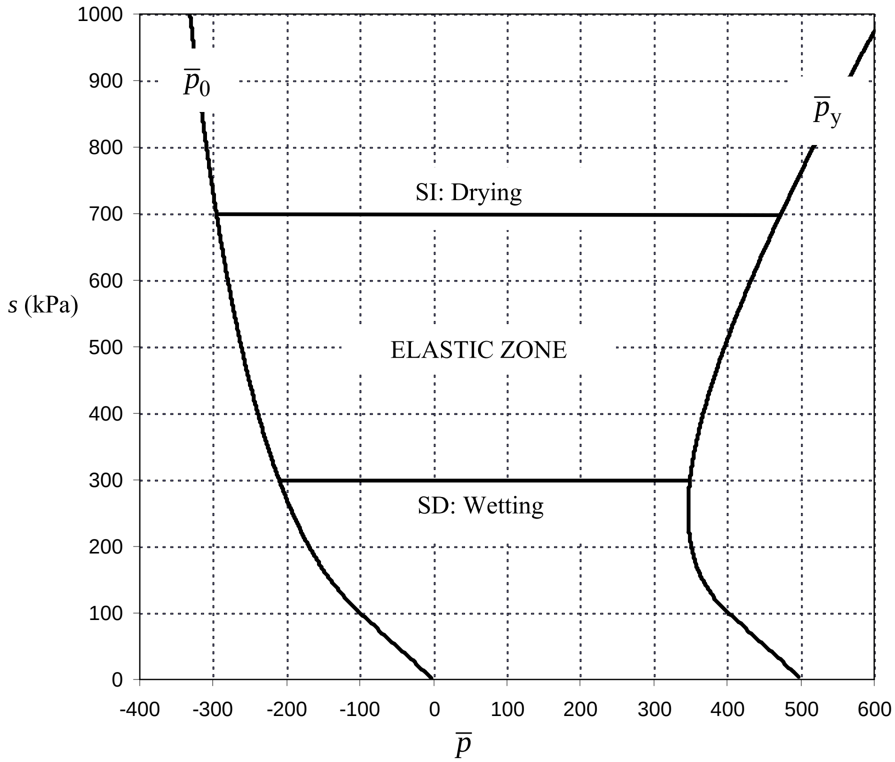


Figure 10. Elastic zone enclosed by the yield surface and the drying and wetting surfaces.

primarily through its influence on the air entry suction and the rate of desaturation (Vanapalli *et al*., 1999). This effect is not considered in this paper.

Hysteresis between the drying and wetting soil-water characteristic curves is usually considered to be too important to ignore. Therefore, a wetting curve must be added and this curve is controlled by the water entry value s_{we} and has a similar slope to that obtained for drying, λ_{ws} (see Figure 9). A series of parallel lines having a slope κ_{ws}, are used to represent recoverable changes in S_r between the drying (desorption) and the wetting (adsorption) curves. These curves are called "scanning curves". For the purpose of this study, the slope of the scanning curve is assumed to be identical to the slope of the drying curve for suctions below the air entry value and suctions above the residual value. The slope of the wetting curve for suctions above the water entry value is also assumed to be κ_{ws} (see Figure 9). The simplifications adopted here are similar to those proposed in the model by Wheeler *et al*. (2003). In the simplified model, the maximum suction that corresponds to full saturation is the saturation suction (s_{sa}), not the air entry value (s_{ae}).

Hysteresis of soil-water characteristic curves can also be explained within the same framework of elasto-plasticity (Sheng *et al*., 2004). Under such a framework, an unsaturated state always lies within the main drying and wetting curves. Drying or wetting from within the hysteresis loops will only cause recoverable water content changes until the suction reaches the main drying or wetting curve. Once soil suction reaches the main drying or wetting curve, further drying or wetting will cause irrecoverable water content changes. Therefore, the drying and wetting curves define the boundaries of recoverable water content change and are similar to the normal compression line. The scanning curves define the recoverable water content change and are similar to the unloading-reloading line. On the $\overline{p}-s$ plane, two additional boundaries can be added, representing the main drying and wetting curves, respectively (Figure 10).

7 Incremental stress-strain relations

The ultimate goal of constitutive modeling is to develop an incremental stress-strain relationship that can be implemented in a numerical method such as the finite element method to solve boundary value problems. For unsaturated soils, these incremental relations can be written in the following form

$$\begin{pmatrix} \mathrm{d}\bar{\boldsymbol{\sigma}} \\ \mathrm{d}s \end{pmatrix} = \begin{pmatrix} \mathbf{D}^{\mathrm{ep}} & \mathbf{W}^{\mathrm{ep}} \\ \mathbf{R} & G \end{pmatrix} \begin{pmatrix} \mathrm{d}\boldsymbol{\varepsilon} \\ \mathrm{d}\theta \end{pmatrix} \tag{31}$$

as in Sheng, Fredlund and Gens (2006), or in the following form

$$\begin{pmatrix} \mathrm{d}\boldsymbol{\sigma}' \\ \mathrm{d}s \end{pmatrix} = \begin{pmatrix} \mathbf{D}^{\mathrm{ep}} & \mathbf{W}^{\mathrm{ep}} \\ \mathbf{R} & G \end{pmatrix} \begin{pmatrix} \mathrm{d}\boldsymbol{\varepsilon} \\ \mathrm{d}S_{\mathrm{r}} \end{pmatrix} \tag{32}$$

as in Sheng *et al.* (2004), depending the stress variables chosen. It is noted that in the displacement finite element method, the pore pressures and displacements are first solved from equilibrium and continuity equations. Therefore, the strain and suction increments are known, and the stress and water content increments must be found from the constitutive equations. In such a context, equations (31) and (32) have to be reformulated so that all known increments are on one side of the equation.

The model by Sheng, Fredlund and Gens (2006) can be used to demonstrate the derivation of the incremental stress-strain equation. The Modified Cam Clay model is used as the base model for saturated soils. The yield function then takes the form of

$$f = q^2 - M^2(\bar{p} - \bar{p}_0)(\bar{p}_{\mathrm{y}} - \bar{p}) \equiv 0 \tag{33}$$

The consistency condition for this yield function can be written as:

$$\mathrm{d}f = \left(\frac{\partial f}{\partial \bar{\boldsymbol{\sigma}}}\right)^{\mathrm{T}} \mathrm{d}\bar{\boldsymbol{\sigma}} + \frac{\partial f}{\partial \bar{p}_0}\frac{\partial \bar{p}_0}{\partial s}\mathrm{d}s + \frac{\partial f}{\partial \bar{p}_{\mathrm{y}}}\frac{\partial \bar{p}_{\mathrm{y}}}{\partial s}\mathrm{d}s + \frac{\partial f}{\partial \bar{p}_{\mathrm{y}}}\frac{\partial \bar{p}_{\mathrm{y}}}{\partial \varepsilon_{\mathrm{v}}^{\mathrm{p}}}\mathrm{d}\varepsilon_{\mathrm{v}}^{\mathrm{p}} \equiv 0 \tag{34}$$

The strain decomposition and the flow rule can be written as:

$$\mathrm{d}\boldsymbol{\varepsilon} = \mathrm{d}\boldsymbol{\varepsilon}^{\mathrm{e}} + \mathrm{d}\boldsymbol{\varepsilon}^{\mathrm{p}} = \mathrm{d}\boldsymbol{\varepsilon}^{\mathrm{e}} + \dot{\Lambda}\frac{\partial g}{\partial \bar{\boldsymbol{\sigma}}} \tag{35}$$

where *g* is the plastic potential function, and $\dot{\Lambda}$ is the plastic multiplier to be solved from equation (34).

The elastic stress-suction-strain relation can be written as:

$$\begin{aligned} &\mathrm{d}\boldsymbol{\varepsilon}^{\mathrm{e}} = \left(\mathbf{D}^{\mathrm{e}}\right)^{-1}\mathrm{d}\bar{\boldsymbol{\sigma}} + \left(\underline{\mathbf{W}}^{\mathrm{e}}\right)^{-1}\mathrm{d}s, \qquad \text{or} \\ &\mathrm{d}\bar{\boldsymbol{\sigma}} = \mathbf{D}^{\mathrm{e}}\mathrm{d}\boldsymbol{\varepsilon}^{\mathrm{e}} - \mathbf{D}^{\mathrm{e}}\left(\underline{\mathbf{W}}^{\mathrm{e}}\right)^{-1}\mathrm{d}s = \mathbf{D}^{\mathrm{e}}\mathrm{d}\boldsymbol{\varepsilon}^{\mathrm{e}} - \mathbf{W}^{\mathrm{e}}\mathrm{d}s \end{aligned} \tag{36}$$

where $\mathbf{D}^{\mathrm{e}}$ is the elastic stress-strain stiffness matrix, $\underline{\mathbf{W}}^{\mathrm{e}}$ is the elastic suction-strain vector, and $\mathbf{W}^{\mathrm{e}} = \mathbf{D}^{\mathrm{e}}\left(\underline{\mathbf{W}}^{\mathrm{e}}\right)^{-1}$.

The plastic multiplier can be found by replacing equations (35) and (36) into equation (34):

$$\dot{\Lambda} = \frac{\left(\dfrac{\partial f}{\partial \bar{\boldsymbol{\sigma}}}\right)^{\mathrm{T}} \mathbf{D}^{\mathrm{e}}\,\mathrm{d}\boldsymbol{\varepsilon} + \left(\dfrac{\partial f}{\partial \bar{p}_0}\dfrac{\partial \bar{p}_0}{\partial s} + \dfrac{\partial f}{\partial \bar{p}_{\mathrm{y}}}\dfrac{\partial \bar{p}_{\mathrm{y}}}{\partial s} - \left(\dfrac{\partial f}{\partial \bar{\boldsymbol{\sigma}}}\right)^{\mathrm{T}} \mathbf{W}^{\mathrm{e}}\right)\mathrm{d}s}{\left(\dfrac{\partial f}{\partial \bar{\boldsymbol{\sigma}}}\right)^{\mathrm{T}} \mathbf{D}^{\mathrm{e}}\dfrac{\partial g}{\partial \bar{\boldsymbol{\sigma}}} - \dfrac{\partial f}{\partial \bar{p}_{\mathrm{y}}}\dfrac{\partial \bar{p}_{\mathrm{y}}}{\partial \varepsilon_{\mathrm{v}}^{\mathrm{p}}}\dfrac{\partial g}{\partial \bar{p}}} \tag{37}$$

The stress-strain relation is then derived:

$$\mathrm{d}\overline{\boldsymbol{\sigma}} = \mathbf{D}^{\mathrm{e}}\mathrm{d}\boldsymbol{\varepsilon} - \frac{\mathbf{D}^{\mathrm{e}}\dfrac{\partial g}{\partial \overline{\boldsymbol{\sigma}}}\left(\dfrac{\partial f}{\partial \overline{\boldsymbol{\sigma}}}\right)^{\mathrm{T}}\mathbf{D}^{\mathrm{e}}\,\mathrm{d}\boldsymbol{\varepsilon} + \mathbf{D}^{\mathrm{e}}\dfrac{\partial g}{\partial \overline{\boldsymbol{\sigma}}}\left(\dfrac{\partial f}{\partial \overline{p}_0}\dfrac{\partial \overline{p}_0}{\partial s} + \dfrac{\partial f}{\partial \overline{p}_{\mathrm{y}}}\dfrac{\partial \overline{p}_{\mathrm{y}}}{\partial s} - \left(\dfrac{\partial f}{\partial \overline{\boldsymbol{\sigma}}}\right)^{\mathrm{T}}\mathbf{W}^{\mathrm{e}}\right)\mathrm{d}s}{\left(\dfrac{\partial f}{\partial \overline{\boldsymbol{\sigma}}}\right)^{\mathrm{T}}\mathbf{D}^{\mathrm{e}}\dfrac{\partial g}{\partial \overline{\boldsymbol{\sigma}}} - \dfrac{\partial f}{\partial \overline{p}_{\mathrm{y}}}\dfrac{\partial \overline{p}_{\mathrm{y}}}{\partial \varepsilon_{\mathrm{v}}^{\mathrm{p}}}\dfrac{\partial g}{\partial \overline{p}}} \tag{38}$$

The suction-water content relation is given by

$$\mathrm{d}\theta = -\lambda_{\mathrm{ws}} n \frac{\mathrm{d}s}{s} + S_{\mathrm{r}}\mathrm{d}\varepsilon_{\mathrm{v}} = -\lambda_{\mathrm{ws}} n \frac{\mathrm{d}s}{s} + S_{\mathrm{r}}\,\mathbf{m}^{\mathrm{T}} \cdot \mathrm{d}\boldsymbol{\varepsilon} \tag{39}$$

where $\mathbf{m}^{\mathrm{T}}$=(1, 1, 1, 0, 0, 0). The slope λ_{ws} should be replaced by κ_{ws} for suction changes along scanning curves.

Therefore, using the following notations

$$\mathbf{D}^{\mathrm{ep}} = \mathbf{D}^{\mathrm{e}} - \frac{\mathbf{D}^{\mathrm{e}}\dfrac{\partial g}{\partial \overline{\boldsymbol{\sigma}}}\left(\dfrac{\partial f}{\partial \overline{\boldsymbol{\sigma}}}\right)^{\mathrm{T}}\mathbf{D}^{\mathrm{e}}}{\left(\dfrac{\partial f}{\partial \overline{\boldsymbol{\sigma}}}\right)^{\mathrm{T}}\mathbf{D}^{\mathrm{e}}\dfrac{\partial g}{\partial \overline{\boldsymbol{\sigma}}} - \dfrac{\partial f}{\partial \overline{p}_{\mathrm{y}}}\dfrac{\partial \overline{p}_{\mathrm{y}}}{\partial \varepsilon_{\mathrm{v}}^{\mathrm{p}}}\dfrac{\partial g}{\partial \overline{p}}}, \qquad G = -{}^{\lambda_{\mathrm{ws}} n}\!/_{s} \quad \text{or} \quad G = -{}^{\kappa_{\mathrm{ws}} n}\!/_{s}$$

$$\mathbf{W}^{\mathrm{ep}} = -\frac{\mathbf{D}^{\mathrm{e}}\dfrac{\partial g}{\partial \overline{\boldsymbol{\sigma}}}\left(\dfrac{\partial f}{\partial \overline{p}_0}\dfrac{\partial \overline{p}_0}{\partial s} + \dfrac{\partial f}{\partial \overline{p}_{\mathrm{y}}}\dfrac{\partial \overline{p}_{\mathrm{y}}}{\partial s} - \left(\dfrac{\partial f}{\partial \overline{\boldsymbol{\sigma}}}\right)^{\mathrm{T}}\mathbf{W}^{\mathrm{e}}\right)}{\left(\dfrac{\partial f}{\partial \overline{\boldsymbol{\sigma}}}\right)^{\mathrm{T}}\mathbf{D}^{\mathrm{e}}\dfrac{\partial g}{\partial \overline{\boldsymbol{\sigma}}} - \dfrac{\partial f}{\partial \overline{p}_{\mathrm{y}}}\dfrac{\partial \overline{p}_{\mathrm{y}}}{\partial \varepsilon_{\mathrm{v}}^{\mathrm{p}}}\dfrac{\partial g}{\partial \overline{p}}}, \qquad \mathbf{R} = S_{\mathrm{r}}\,\mathbf{m}$$

the final incremental stress-strain relationship can be written:

$$\begin{pmatrix} \mathrm{d}\overline{\boldsymbol{\sigma}} \\ \mathrm{d}\theta \end{pmatrix} = \begin{pmatrix} \mathbf{D}_{\mathrm{ep}} & \mathbf{W}_{\mathrm{ep}} \\ \mathbf{R} & G \end{pmatrix} \begin{pmatrix} \mathrm{d}\boldsymbol{\varepsilon} \\ \mathrm{d}s \end{pmatrix} \tag{40}$$

where $\mathbf{D}^{\mathrm{ep}}$ is a 6×6 matrix, **R** is a row vector of 6 elements, $\mathbf{W}^{\mathrm{ep}}$ is a column vector of 6 elements, and *G* is a scalar. The rate of soil suction is kept on the right-hand side as the strain rate, in consistent with the displacement finite element method where pore pressures and displacements are first solved from equilibrium and continuity equations. The incremental stress-strain relationship defined by equation (40) can be implemented into the finite element method to solve boundary value problems. The implementation follows Sheng *et al*. (2003a, 2003b). Due to the non-convexity of the yield surface, special techniques are required regarding the integration of the rate equation.

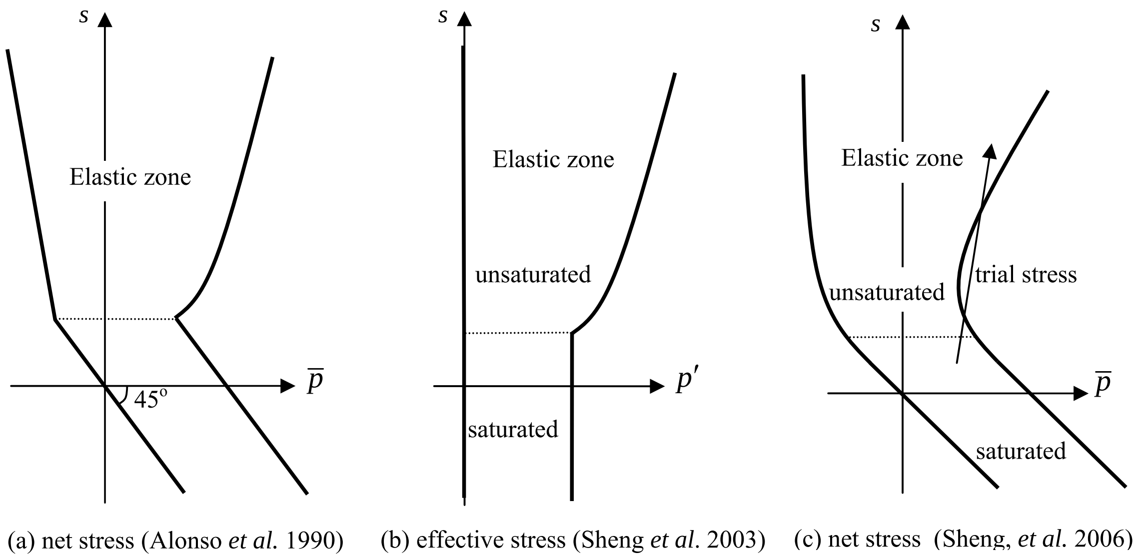


Figure 11. Non-convexity of yield surfaces for unsaturated soils in a suction-stress space.

8 Finite Element Implementation

The main challenge in implementing an unsaturated soil model into finite element code arises from the non-convexity of the yield surface around the transition between saturated and unsaturated states. The non-convexity exists irrespective of the stress variables used in the model and is demonstrated in Figure 11.

For given strain and suction increments, the current stress state and internal variables must be updated in accordance with equation (31) or (32). This update is generally carried out using numerical stress integration schemes. Both implicit and explicit schemes have been used to integrate elastoplastic models. Implicit schemes, where all gradients are estimated at an advanced stress state, cannot be used for elastoplastic models with non-convex yield surfaces, because the extrapolated gradients cannot be determined due to the uncertainty of whether an advanced position is inside or outside the yield surface. On the other hand, explicit schemes can proceed in an incremental fashion, but require the intersection between the current yield surface and an elastic trial stress path to be determined.

A key issue in integrating the incremental stress-strain relationships using an explicit method is thus to find the intersection between the elastic trial stress and the current yield surface. The most complicated situation occurs when the yield surface is crossed more than once. However, it is not possible to know *a priori* how many times the yield surface is crossed, because the size of the yield surface will change after the first intersection due to hardening. Therefore, for non-convex yield surfaces, the key task is to find the very first intersection for any possible path.

In order to determine whether the yield surface is crossed, a secant trial stress increment can be computed, based on an elastic stress-suction-strain relationship. This elastic trial stress is given as follows:

$$\Delta\boldsymbol{\sigma}^{\mathrm{tr}} = \mathbf{D}^{\mathrm{e}} : \Delta\boldsymbol{\varepsilon} + \mathbf{W}^{\mathrm{e}}\Delta s \tag{41}$$

where the stress is either the net stress or effective stress (depending on the model), $\mathbf{D}^{\mathrm{e}}$ is the fourth order elastic stiffness tensor (in tensor notation) and $\mathbf{W}^{\mathrm{e}}$ is a second order tensor defined according to a specific law for unsaturated soils. For saturated soil models, the term $\mathbf{W}^{\mathrm{e}}\Delta s$ depends on the stress variables used. If the effective stress is used, the term $\mathbf{W}^{\mathrm{e}}\Delta s$ becomes zero and can be disregarded. On the other hand, if the net stress is used, the term $\mathbf{W}^{\mathrm{e}}\Delta s$ becomes $-\mathbf{m}\Delta u_{\mathrm{w}}$, where **m** is the second order identity tensor and u_{w} the pore water pressure.

In equation (41), $\Delta\boldsymbol{\varepsilon}$ is the strain increment provided from the finite element routines prior to the computation of the residuals between internal and external forces. For unsaturated soils, the increment of suction Δs is also input for the stress-update algorithm. If the elastic modulus is linear, i.e. it is independent of the stresses, suction and internal variables, it is trivial to compute the elastic trial increment. Otherwise, for some non-linear relations, a secant analytical modulus may be considered.

Finding the intersection between the elastic trial stress increment and the current yield surface can be cast into the problem of finding multiple roots of a nonlinear equation. $f_\alpha(\alpha) = 0$. The roots (α) must be computed inside the interval $[0,1]$. As this function involves the evaluation of the yield function along the strain and suction paths, it is given as

$$f_\alpha(\alpha) = f(\boldsymbol{\sigma}_\alpha, s_\alpha, z_k) \tag{42}$$

where $f(\boldsymbol{\sigma}, s, z_k)$ is the yield function, z_k indicates a set of internal variables and the intermediate stress-suction states σ_α and s_α are calculated according to

$$\boldsymbol{\sigma}_\alpha = \boldsymbol{\sigma}_{\text{current}} + \alpha\Delta\boldsymbol{\sigma}^{\text{tr}} \quad \text{and} \quad s_\alpha = s_{\text{current}} + \alpha\Delta s \tag{43}$$

in which $\boldsymbol{\sigma}_{\text{current}}$ and s_{current} are the current stress and suction states. Note that in equation (42) the internal variables z_k are kept constant during the solution for the intersection. These variables only change during hardening/softening when a portion of the trial stress-suction path is located outside the yield surface.

The technique proposed herein follows the Kronecker-Picard (KP) formula for the determination of the number of roots of a nonlinear equation (Kavvadias *et al.*, 1999). This formula, given by

$$N = \frac{-\gamma}{\pi}\int_a^b \frac{f_\alpha(x)h_\alpha(x) - g(x)^2}{f_\alpha(x)^2 + \gamma^2 g_\alpha(x)^2}\,\mathrm{d}x + \frac{1}{\pi}\arctan\left\{\frac{\gamma\left(\left[f_\alpha(a)g_\alpha(b) - f_\alpha(b)g_\alpha(a)\right]\right)}{f_\alpha(a)f_\alpha(b) + \gamma^2 g_\alpha(a)g_\alpha(b)}\right\} \tag{44}$$

requires that $f_\alpha(\alpha)$ must be continuously or piecewise differentiable to the second order for values of α from *a* to *b*. In equation (44), g_α and h_α represent the first and second derivatives of the function f_α with respect to α, respectively, and γ is a small positive constant which does not affect the results computed with the KP formula (Kavvadias *et al.*, 1999). The first and second derivative of f_α with respect to α can be directly determined as follows:

$$g_\alpha(\alpha) = \frac{\partial f_\alpha}{\partial \alpha} = \frac{\partial f_\alpha}{\partial \boldsymbol{\sigma}_\alpha} : \frac{\mathrm{d}\boldsymbol{\sigma}_\alpha}{\mathrm{d}\alpha} + \frac{\partial f_\alpha}{\partial s_\alpha}\frac{\mathrm{d}s_\alpha}{\mathrm{d}\alpha} = \left.\frac{\partial f}{\partial \boldsymbol{\sigma}}\right|_\alpha : \Delta\boldsymbol{\sigma}^{\text{tr}} + \left.\frac{\partial f}{\partial s}\right|_\alpha \Delta s \tag{45}$$

$$h_\alpha(\alpha) = \frac{\partial^2 f_\alpha}{\partial \alpha^2} = \Delta\boldsymbol{\sigma}^{\text{tr}} : \left.\frac{\partial^2 f}{\partial\boldsymbol{\sigma}\partial\boldsymbol{\sigma}}\right|_\alpha : \Delta\boldsymbol{\sigma}^{\text{tr}} + 2\Delta\boldsymbol{\sigma}^{\text{tr}} : \left.\frac{\partial^2 f}{\partial\boldsymbol{\sigma}\partial s}\right|_\alpha \Delta s + \left.\frac{\partial^2 f}{\partial s^2}\right|_\alpha \Delta s^2 \tag{46}$$

The number of roots estimated according to equation (44) is used to divide the interval of α into subintervals until each subinterval contains at most one root. First, *N* is computed for the interval [*a*, *b*]. If *N* is larger than one, the interval [*a*, *b*] is divided into two equal subintervals, $\left[a, (a+b)/2\right]$ and $\left[(a+b)/2, b\right]$. The number of roots for each subinterval is then computed and any subinterval that contains more than one root is further divided into two equal sub- subintervals. This process continues until each subinterval contains at most one root. As shown by Kavvadias *et al*. (1999), the usage of equal-size intervals (equiprobable parts) is not much worse than an algorithm which would consider the statistical distribution of the roots inside $[a,b]$, such as the algorithm presented in Kavvadias *et al.* (1999).

Once the roots are bracket, the solution of each root can be found by using existing numerical methods such as the Newton-Raphson method. It should be noted that the Newton-Raphson method, although fast, may not converge in some circumstances because it does not constrain the solution to lie within specified bounds. Therefore, more advanced methods can be used. For example, the Pegasus method used in Sloan *et al*. (2001) is robust and competitively fast. The method by Brent (1971) provides another attractive alternative. The Brent method does not use any derivative, does not require initial guesses and guarantees the convergence as long as the values of the function are computable within a given region containing a root. This characteristic of the Brent method is due to the combination of the bisection method, the secant method and inverse quadratic interpolation. Therefore, it has the reliability of the bisection method and the efficiency of the less reliable secant method or inverse quadratic interpolation.

The evaluation of the integral in equation (44) with the KP formula is generally not trivial and so a numerical integration or quadrature technique has to be used. For example, the Gauss-Legendre method (Forsythe *et al*., 1990) can be used here. In addition, for highly non-linear yield functions, an adaptive integration scheme may also have to be used. Detailed information in this regard can be found in Pedroso *et al* (2007) and Sheng *et al.* (2007a).

9 Conclusions

A number of conclusions can be drawn from this study:

1. The use of an effective stress for unsaturated soils can lead to a smooth transition between saturated and unsaturated states. However, the key issue is that the effective stress usually becomes a material property and even depends on the material state when the soil becomes unsaturated. Therefore, a constitutive relation established in such an effective stress space is less meaningful, since the stress space is constantly changing with the material state.

2. The use of independent stresses for modelling unsaturated soils often leads to discontinuous models at the transition between saturated and unsaturated states. However, this problem can be avoided if the volumetric stress – strain model is continuous across the transition.

3. Most existing models have difficulties in modelling soils dried from slurry. The stress state during the drying of a slurry soil should always be on the current yield surface. The normal compression lines under constant suctions for soils dried from slurry are naturally curved and this curvature is not due to over-consolidation.

4. Most elastoplastic models have embodied shear strength criteria. For example in Barcelona Basic model, the friction angle ϕ^{b} due to suction is assumed to be constant. In the model by Sheng, Fredlund and Gens (2006), the friction angle ϕ^{b} is a function of suction and the air entry value.

5. Hysteresis in the soil-water characteristic curves can be formulated in the same framework of elastoplasticity, which leads to a consistent formulation of stress – strain and suction – saturation relations.

6. Unsaturated soil models have inevitably non-convex yield surfaces at the transition between saturated and unsaturated states. This non-convexity can significantly complicate the implementation of these models into finite element codes.

7. An explicit stress integration scheme incorporating an efficient root search algorithm can be used integrate an unsaturated soil model with a non-convex yield surface.

10 References

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