Independent Roles of the Stress State Variables on Volume–Mass Constitutive Relations

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Summary. This paper investigates the individual character of the two independent stress tensors and the effect of each on the volume-mass constitutive relations for unsaturated soils. Cases involving soils with varying plasticity and stress history are presented. A model for void ratio (or specific volume) behaviour has been developed within the context of independent stress state variables. Theoretical and experimental results show that changes in the net total stress tensor produce distinctly independent volume change behaviour from changes in matric suction even when the net total stress changes are isotropic. Stress history is reflected differently on the volume change constitutive relations depending upon which stress tensor is changed. Likewise, the water content constitutive relationship is affected in a distinctively different manner depending upon which stress tensor is changed.

1 Introduction

Independent stress state variables were introduced in the form of two tensors in the 1970s to describe the stress state for unsaturated soils (Fredlund and Morgenstern 1977). Since that time there have been numerous attempts to develop elasto-plastic models to describe the behaviour of unsaturated soils (Alonso et al. 1990, Wheeler and Sivakumar 1995, Blatz and Graham 2003). Generally, simplistic assumptions have been made with regard to the relationship between water content changes and volume changes in an unsaturated soil.

A volume-mass constitutive model in terms of two independent stress state variables is presented in this paper. The model is capable of predicting volume change and water content independently. The hysteretic nature of the soil-water characteristic curve is also taken into account. The model is proposed for only isotropic condition (i.e., ignores the existence of the shear stress).

2 Terminology

Deformation of a soil mass is directly related to a change in the volume of pores in the soil. The pore-size distribution curve (PDC) of a soil at any stress state provides information regarding both the total volume and the volume of water in the soil. Therefore, the prediction of the PDC is important in a volume-mass constitutive model.

The PDC is a function of soil suction and net mean stress (Pham 2005). In the proposed model, a reference pore-size distribution is first selected. The stress-strain relationship for the soil structure surrounding a pore group is then presented and followed by the calculation for pore groups along the reference pore-size distribution curve.

3 Theory

3.1 Stress State Variables

The proposed model makes use of two stress state variables (Fredlund and Morgenstern 1977): net mean stress $p = (\sigma_1 + \sigma_2 + \sigma_3)/3 - u_a$ and soil suction $\psi = (u_a - u_w)$. The reference stress state is chosen to be a net mean stress = 1 kPa and soil suction = 0 kPa (i.e., equivalent with effective stress = 1 kPa at saturation). The void ratio, e, of the soil is the primarily variable used to represent the overall volume of the soil. The gravimetric water content, w, of the soil is the primary variable used to represent the amount of water in the soil.

3.2 3.2 Reference Pore-Size Distributions

The PDC corresponding to the completely dry condition obtained from an initial slurry soil (i.e., at 10^6 kPa on the initial drying process of the slurry soil) provides a meaningful reference state. In this model, the *reference pore-size distribution curve* of a soil is defined as the PDC corresponding to the completely dry condition of a soil.

A soil has two reference pore-size distribution curves at completely dry conditions (i.e., with respect to drying and wetting suctions). The *reference drying DPC* of a soil provides information regarding the air entry value and the distribution in volume of pores in the soil. Similarly, the *reference wetting PDC* of a soil provides information regarding the water entry value and the distribution in volume of pores in the soil.

3.3 Basic Assumptions

Six assumptions based on the results of previous studies were adopted for the proposed constitutive model.

- *Assumption #1*: A particular pore under consideration in the soil has only two states; namely, i) the pore is filled with water; or ii) the pore is dry.
- *Assumption #2*: Each water-filled pore in the soil also has two indices, namely: i) virgin compression index, and ii) unloading-reloading compression index.
- *Assumption #3*: There are two types of pores: i) compressible pores and ii) non-compressible pores. The compressible pores are relatively large pores and the non-compressible pores are relatively small-interconnected pores.
- *Assumption #4*: Virgin and unloading-reloading compression indices for a pore in the soil are proportional to the volume of the pore at the reference stress state.
- *Assumption #5*: Pores are deformed, and water is absorbed and drained independently.
- *Assumption #6*: Air-filled pores are incompressible.

3.4 Mathematical Formulation

The initial drying SWCC from the slurry of a significant volume change soil can be best-fitted using the following empirical equation:

$$w(\psi) = \left(\left[w_{\mathrm{sat}} - \frac{C_c}{G_s}\log\psi - w_r\right]\frac{a}{\psi^b + a} + w_r\right)\left(1 - \frac{\ln\left[1 + \frac{\psi}{\psi_r}\right]}{\ln\left[1 + \frac{10^6}{\psi_r}\right]}\right) \tag{1}$$

where C_c = virgin compression index of the soil; G_s = Particles specific gravity; w_r = curve-fitting parameter represents the residual water content; w_{sat} = curve-fitting parameter represents the water content at reference stress state; and a, b = curve-fitting parameters. The virgin compression index of the group of pores having air entry value of ψ on the reference pore-size distribution curve can be calculated as follows:

$$C_c^p(\psi) = -\frac{C_c ab\ln(10)\psi^b}{[\psi^b + a]^2}\,. \tag{2}$$

Similarly, the unloading-reloading compression index of the group of pores having air entry value of ψ on the reference pore-size distribution curve can be calculated as follows:

$$C_s^p(\psi) = -\frac{C_s ab\ln(10)\psi^b}{[\psi^b + a]^2}\,. \tag{3}$$

Comparing the strain of a pore that is dried under zero net mean stress and a pore that is dried under a yield stress, p_y, and then dried under a constant net mean stress, p, the equation for the effect of the net mean stress on the air entry value can be written:

$$\frac{\psi_{\mathrm{ae}}}{\psi_{\mathrm{ae}}(p, p_y)} = 1 - \eta\frac{[(C_c - C_s)\log p_y + C_s\log(\psi_{\mathrm{ae}} + p) - C_c\log(\psi_{\mathrm{ae}})]}{3[e_{\mathrm{sat}} - C_c\log\psi_{\mathrm{ae}} - w_r G_s]} \tag{4}$$

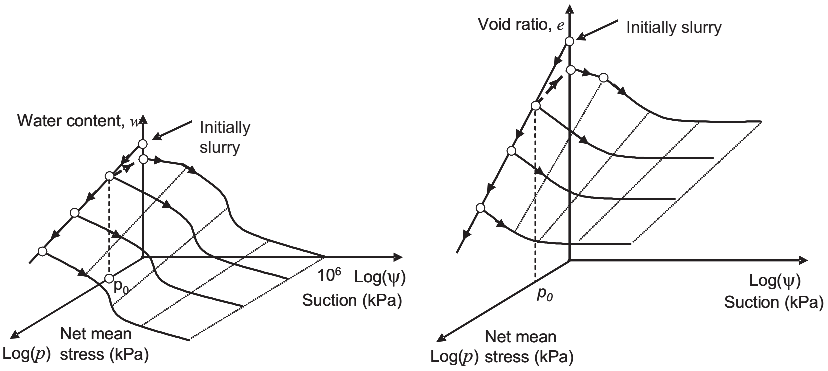
where $\psi_{\mathrm{ae}} =$ water entry value of the pore having zero yield stress and wetting under zero net mean stress; $\psi_{\mathrm{ae}}(p, p_y) =$ air entry value of the pore when yield stress is equal to p_y and drying under a net mean stress of p; $\eta =$ pore-shape parameter represents the effect of the net mean stress to the change in the diameter of a pore. Similarly, the equation for the effect of the net mean stress to the water-entry value of the pore can be obtained.

The yield stress of the soil structure surrounding a water-filled pore is considered to be maximum effective stress that ever acted on the pore. If a pore is dried under zero net mean stress, the yield stress of the pore is equal to the air entry value of the pore. If a pore is dried under a constant net mean stress of, p, the yield stress of the pore can be calculated:

$$p_y = p + \psi(p, p + \psi_{\mathrm{ae}}) \tag{5}$$

where $\psi(p, p + \psi_{\mathrm{ae}})$ can be calculated using Eq. (4). When a pore is filled with air, the pore is incompressible (i.e., assumption #6); therefore, yield stress of the soil structure surrounding the pore does not change with net mean stress and soil suction. The two constitutive equations for the volume-mass constitutive surfaces follow the stress paths shown in Fig. 1. The constitutive surfaces correspond to a slurry soil that is initially loaded to a net mean stress, p_0 at zero soil suction and then dried.

Equations have been written for the water content and the void ratio constitutive surfaces (see Pham 2005). The equation for the degree of saturation surface can be derived from the equations for the gravimetric water content surface and void ratio surface.



a) Water content

b) Void ratio

Fig. 1. Schematic illustration of the volume-mass constitutive surfaces of an initially slurred specimen that is dried under various constant net mean stresses

3.5 Hysteresis Model for SWCC

A hysteresis model is developed using two one-dimensional pore-size distribution functions (i.e., wetting and drying pore-size distributions). Scanning curves are horizontal on degree of saturation SWCC plots (similar to the Wheeler et al. (2003) model). This means that each group of pores has a unique relationship between the drying and wetting suction. Therefore, there is a relationship between the wetting reference DPC and the drying reference DPC.

3.6 Model Parameters

The required data for calibration can be described as follows:

- the initial drying SWCC of the initially slurry soil specimen,
- pore-shape parameter, η,
- the parameters for the hysteretic nature of the SWCC of the soil (Pham et al. 2005),
- the compression indices of the soil.

4 Presentation of the Model Prediction

The application of the proposed volume-mass constitutive model is presented for an artificial silt. The soil properties of the artificial silt are shown in Table 1. The initial drying SWCC for the silt is shown in Fig. 2.

Table 1. Soil properties for the artificial silt

Soil-water characteristic curve				Hysteresis parameters			Compression indices		
w_{sat}	a	b	w_r	D_{SL}	R_{SL}	β	C_c	C_s	G_s
0.45	200,000	2.5	0.08	0.35	1.5	0.1	0.2	0.04	2.7

ht The predicted shrinkage curve for the artificial silt shown in Fig. 3 has a reasonable shape. At high water contents (i.e., almost 100% degree of saturation), the shrinkage curve is a 45 degree line. The shrinkage curves are approximately horizontal at low water contents.

Model predictions for the artificial silt when following a complex stress path (Fig. 4) are presented in Fig. 5. Figure 5**a** shows that the model can take into account plastic volume change along the initial drying process (i.e., drying the soil to 100 kPa). The collapsible behavior of the unsaturated silt during a

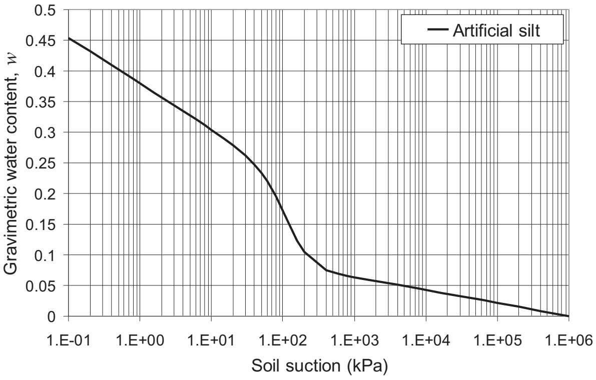


Fig. 2. Initial drying Soil-Water Characteristic Curve for artificial silt

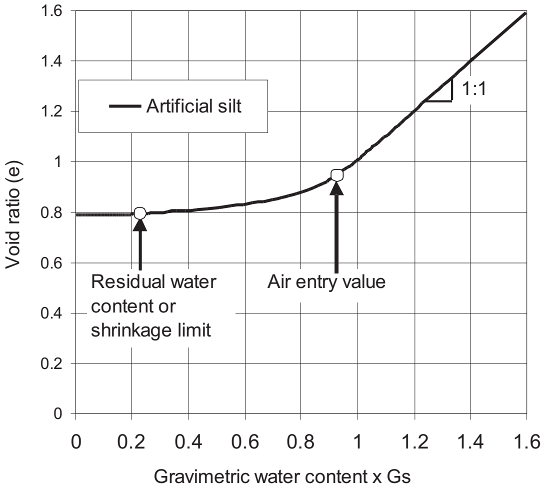
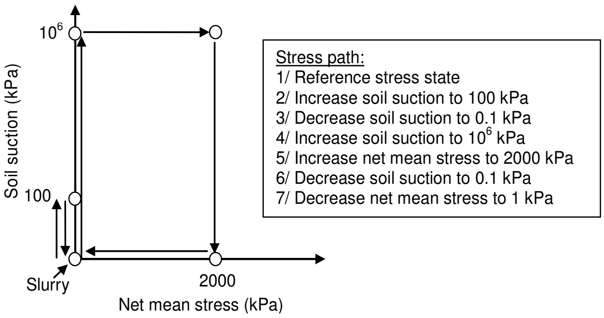
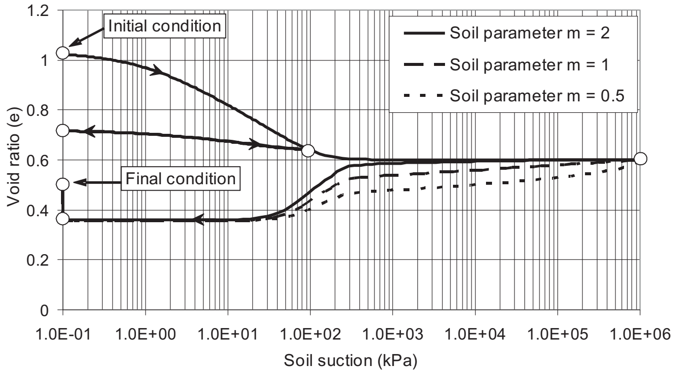


Fig. 3. Predicted shrinkage curve for artificial silt

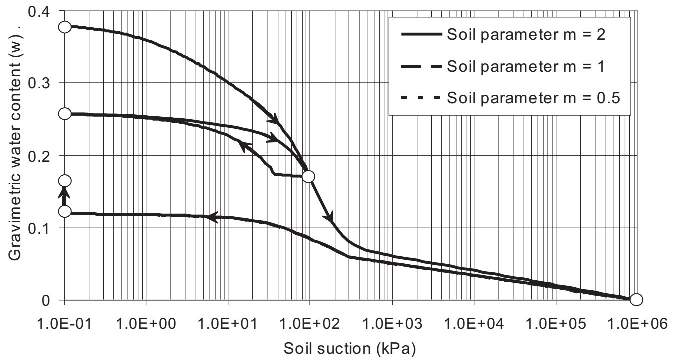


Stress path:
1/ Reference stress state
2/ Increase soil suction to 100 kPa
3/ Decrease soil suction to 0.1 kPa
4/ Increase soil suction to 10^6 kPa
5/ Increase net mean stress to 2000 kPa
6/ Decrease soil suction to 0.1 kPa
7/ Decrease net mean stress to 1 kPa

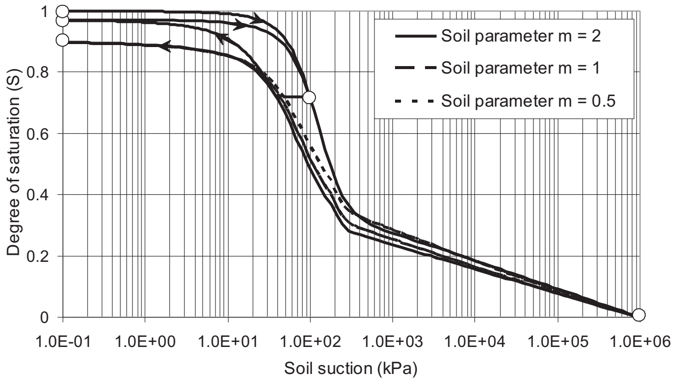
Fig. 4. Stress path followed for artificial silt



a) Void ratio



b) Gravimetric water content



c) Degree of saturation

Fig. 5. Plots of predicted void ratio, gravimetric water content and degree of saturation for artificial silt

wetting process can also be seen in Fig. 5**a**. This mechanical behavior can be observed from many experimental results (Matyas and Radhakrishna 1968, Alonso et al. 1990). The soil parameter m controls the collapsible behavior of the soil. The smaller the value for the soil parameter, m, the earlier the collapse occurs in the soil during the wetting process.

The hysteresis in the gravimetric water content shown in Fig. 5**b** results from: i) plastic volume change and ii) the hysteretic nature of the SWCC. In general, mechanical behaviors of the soil in the model agree well with experimental results of actual soils (Pham 2005).

5 Conclusions and Recommendations

The proposed volume-mass constitutive model is capable of: 1) predicting both volume and water content at all stress states corresponding to a wide variety of stress paths; 2) taking into account the hysteretic nature of the soil-water characteristic curve; and 3) predicting both swelling and collapsible behavior of an unsaturated soil. The model can predict volume-mass constitutive relationships that are stress path dependent. The prediction results appear to be consistent with observed laboratory data.

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