Dynamic programming method in slope stability computations

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Abstract: This article describes a new approach in slope stability computations. The procedure combines an optimization method known as "dynamic programming" with finite element method to search for the critical slip surface and compute the corresponding factor of safety. The critical slip surface is defined as the optimal path through the finite element stress field that yields the minimum value of the "optimal function". A computer program named DYNPROG has been developed based on the analytical procedure described in this article. Comparisons of the present method with numerous limit equilibrium methods of slices showed that the present method provided a more satisfied solution in terms of the location of the critical slip surface as well as the value of the factor of safety.

1 INTRODUCTION

1.1 *Background*

Over the past two decades, there has been variety of slope stability methods proposed in an attempt to make the analysis of slope more complete and comprehensive (Pham, 2002). These methods are considered to be more comprehensive in that the shape and the location of the critical slip surface are parts of the solution for the minimum factor of safety. The critical slip surface and the associated factor of safety are simultaneously determined using mathematical formulations or optimization techniques. Most of these methods have still relied upon one of the limit equilibrium methods of slices when computing the factor of safety (Nguyen, 1985; Li & White, 1985). Some of the proposed methods that use optimization techniques can be considered as *ad hoc* methods that are still inappropriate for practical engineering purposes. There have also been concerns that these methods might yield a local factor of safety rather than a global factor of safety (Greco, 1988).

It is well recognized that the major shortcoming of limit equilibrium methods of slices is the disregard for the actual stresses in the soil. This limitation can be essentially overcome by the introduction of a finite element stress analysis. Limit equilibrium conditions applied to the critical slip surface are more meaningful when the actuating and resisting forces are evaluated using a finite element stress analysis. In 1969, Kulhawy introduced a stability method that combines the theory of limit equilibrium with a finite element stress analysis. Krahn (2001) referred to the article by Fredlund & Scoular (1999) in order to highlight the applicability of finite element method in slope stability analysis. In was also stated that the main disadvantage of using a finite element stress-based approach to the analysis of slopes was not so much a technical issue as it was a lack of experience with the method in geotechnical engineering practice (Krahn, 2001).

1.2 *The use of dynamic programming (DP) technique in slope stability analysis*

DP is an optimization technique, which was first introduced by Bellman (1957). The technique can be considered as an optimal scheme for solving multistage optimization problems (Yamagami and Ueta, 1988) and is applicable only to 'additive functions' (Baker, 1980). In 1980, Baker introduced an approach that utilized the DP technique in slope stability computation. In this approach, Baker (1980) combined an optimization search using the DP technique with the Spencer's (1967) method to locate the critical slip surface and calculate the associate factor of safety. The procedure introduced by Baker was then adopted and improved by several subsequent research workers (Yamagami and Ueta, 1988; Zou *et al.*, 1995).

This article presents a study on the practical application of the DP technique in solving slope stability problems. A new procedure for slope stability analysis that combines a partial general differential equation solver known as FlexPDE with DP technique is introduced. In this approach, a finite element stress analysis using FlexPDE is carried out to obtain a field of finite element stresses within the slope. Afterwards, an optimization search using DP technique will be performed on this stress field to locate the most optimal path that represents the critical slip surface. The corresponding factor of safety with the critical slip surface will be computed using finite element stress values. A computer program named DYNPROG was developed to facilitate the above works.

2 ANALYTICAL PROCEDURE AND APPLICATION

2.1 *DP theory*

Figure 1 presents the analytical scheme for a problem of slope stability analysis. A potential slip path, AB, can be approximated by a series of constitutively linear segments connecting two state points located in two successive stages. The stage-state point system forms a '*search grid*' containing a number of small rectangular elements which are referred to as '*grid element*'. The slip path is assumed to start from and end at initial and final points that are located arbitrarily outside the geometrical boundary of the slope. The intersections of the critical path with the geometrical boundary of the slope are entry and exit points, which bound the actual critical slip surface. The factor of safety, F_s, of the slip surface AB in Fig. 1 is defined as follows:

$$F_s = \frac{\sum_{i=1}^{n} \tau_{f_i} \Delta L_i}{\sum_{i=1}^{n} \tau_i \Delta L_i} \tag{1}$$

where n is the number of discrete segments, τ_i, τ_{f_i} and ΔL_i are the shear mobilized, shear strength and the length of the i^{th} segment, respectively. According to Baker (1980), the factor of safety, defined in Eq. [1], can be minimized by the introduction of the '*auxiliary function*', G, defined by Eq. [2] as follows:

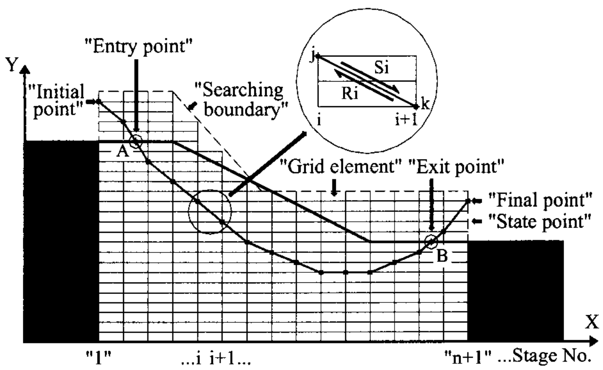

Fig. 1. The analytical scheme of the slope stability analysis using DP technique.

$$G = \sum_{i=1}^{n}(R_i - F_s S_i) = \sum_{i=1}^{n}(\tau_{f_i} - F_s \tau_i)\Delta L_i \tag{2}$$

where terms are defined in Fig. 2. Figure 2 also shows the close-up of a constitutive segment of the slip surface AB. This particular segment is contained by the boundaries of two '*grid elements*'. The resisting and actuating forces on a constitutive segment are assumed to be the summation of the resisting and actuating forces acting on each small division of the segment contained by the boundaries of each element. If we assume that the stresses within an element are constant and are signified by stresses at the center of the element then the auxiliary function can be calculated using σ_x, σ_y and τ_{xy} at the center of each element as follows:

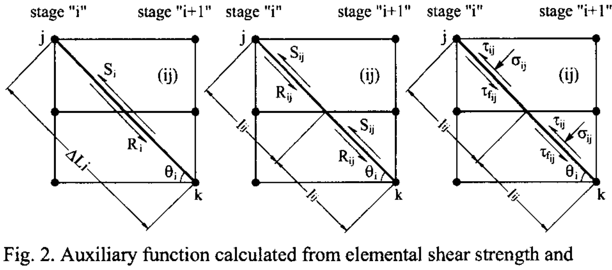

Fig. 2. Auxiliary function calculated from elemental shear strength and shear stresses.

$$G = \sum_{i=1}^{n}\left\{\sum_{ij=1}^{ne}\left[c'_{ij} + \left(\sigma_{ij} - u_a^{ij}\right)\tan\phi'_{ij} + \left(u_a^{ij} - u_w^{ij}\right)\tan\phi_{ij}^{b}\right] l_{ij} - F_s \sum_{ij=1}^{ne}\tau_{ij} l_{ij}\right\} \tag{3}$$

$$\sigma_{ij} = \sigma_x^{ij}\sin^2\theta_i + \sigma_y^{ij}\cos^2\theta_i - \tau_{xy}^{ij}\sin 2\theta_i \tag{4}$$

$$\tau_{ij} = \tau_{xy}^{ij}(\sin^2\theta_i - \cos^2\theta_i) - \frac{\sigma_y^{ij} - \sigma_x^{ij}}{2}\sin 2\theta_i \tag{5}$$

where *ne* is the number of '*grid elements*', *ij*, passed by a constitutive segment. The shear strength of the soil was calculated in accordance with the theory of unsaturated soil mechanics (Fredlund and Rahardjo 1993). In Eq. [2], shear strength parameters are denoted by c'_{ij}, ϕ'_{ij} and ϕ_{ij}^{b}. Pore pressures, u^{ij}, which are obtained from a finite element seepage analysis are used to calculate effective stresses the strength formulation.

Let us introduce an '*optimal function*' obtained at stage [*i*], $H_i(j)$, which is equal to the minimum value of the '*auxiliary function*', G, calculated along the constitutive segment that connects a state point in the initial stage to state point {*j*} in stage [*i*]. According to '*the principle of optimality*' (Bellman, 1957), the optimal function, $H_{i+1}(k)$, obtained at state point {*k*} located on stage [*i+1*] can be calculated as:

$$H_{i+1}(k) = H_i(j) + G_i(j,k) \tag{6}$$

where $G_i(j,k)$ is the '*auxiliary function*' calculated along the constitutive segment that connects state point {*j*} in stage [*i*] to state point {*k*} in stage [*i+1*]. At the initial stage (i.e., outside the slope), the value of the '*optimal function*', $H_1(j)$, is equal to zero, that is:

$$H_1(j) = 0 \qquad j = 1...NP_1 \tag{7}$$

where NP_1 is the number of state points in the initial stage. At the final stage (*i=n+1*), the 'optimal function', $H_{n+1}(k)$, is equal to the minimum value of the summation of the '*auxiliary function*', G, calculated from the initial stage to the final stage:

$$H_{n+1}(k) = H_n(j) + G_n(j,k) \tag{8}$$

$$H_{n+1}(k) = G_{\min} = \min\left[\sum_{i=1}^{n}(R_i - F_s S_i)\right];\ k = 1..NP_{n+1} \tag{9}$$

where NP_{n+1} is the number of state points in the final stage. The '*optimal point*' in the final stage is defined as the point at which, the '*optimal function*' calculated is minimum. From the '*optimal point*' {*k*} in the final stage, the '*optimal point*' {*j*} in the previous stage is also determined. The whole '*optimal path*' defined by connecting '*optimal points*' in every stage will be eventually located by tracing back from the final stage to the initial stage. The factor of safety corresponding to this '*optimal path*' will then be computed using Eq. [1].

It can be inferred that the value of F_s in Eq. [2] has to be assumed prior to the performance of the optimization process. The value of F_s used for the next trial optimization process will be averaged based on the assumed value of F_s and the new F_s computed at the end of the current trial. The optimization process will be stopped when a pre-defined convergence is met. The '*optimal path*' that corresponds to the final value of F_s is the '*critical slip path*' through the slope.

1.2 *Finite element stress analysis*

The general partial differential equation solver known as FlexPDE is a computer program for obtaining numerical solutions to single or coupled sets of partial differential equations. In a plain strain condition, a soil element that is subjected to its body forces has partial differential equations representing the stress balance defined as follows:

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + F_x = 0 \tag{10}$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + F_y = 0 \tag{11}$$

where $\sigma_x,\sigma_y,\tau_{xy},F_x,F_y$ are respectively normal Cartesian stress, shear stress and body forces in x- and y-coordinate directions. Partial differential equations [10] and [11] can be solved using FlexPDE with specified boundary conditions. The stresses are evaluated and stored at Gaussian points over the domain of the problem. It is of interest to design an output grid in FlexPDE that is coincident with the '*search grid*' using in the optimization process since FlexPDE can interpolate and export stresses from Gaussian nodes to any arbitrary grid with coordinates defined by the user. The stresses at the center point of each grid element are interpolated using the shape function (Bathe, 1982). The stress interpolation process must be done prior to the performance of the optimization search.

1.3 *Finite element seepage analysis*

The following differential equation can be used to represent the two-dimensional, steady-state water flow through a non-homogeneous, anisotropic, saturated/unsaturated soil:

$$k_x \frac{\partial^2 h}{\partial x^2} + k_y \frac{\partial^2 h}{\partial y^2} + \frac{\partial k_x}{\partial x}\frac{\partial h}{\partial x} + \frac{\partial k_y}{\partial y}\frac{\partial h}{\partial y} = 0 \qquad (12)$$

The detailed solution of Eq. [12] using FlexPDE was presented in Nguyen (1999). Essentially, the solution for Eq. [12] is the same as that used for solving Eq. [10] and [11].

Pore pressures computed using FlexPDE are also output to the designated grid that is coincident with the output grid of stresses. The assumption by that the pore pressure is constant within a small element, and is signified by the value of the pore pressure at the centre point of that element, is also utilized. The interpolation process using shape functions for interpolating pore pressure at a point from the nodal values is essentially the same as the process described in Section 1.3. Values of pore pressure are used for calculating effective stresses in the strength formulation.

1.4 *Description of the analytical procedure*

An analytical procedure based on the theory presented in preceding sections is developed. The detailed description of the procedure can be found in Pham & Fredlund (2003). A computer program named DYNPROG was also developed based on this procedure to facilitate the analysis. The procedure can be summarized as follows:

(1) Import the output grid with corresponding nodal stresses from FlexPDE. This output grid is coincident with the '*search grid*' using in the optimization process.

(2) Interpolate stresses at the center point of each grid element from nodal stresses using shape functions (Bathe, 1982).

(3) Assume an initial F_s (i.e., F_s =1).

(4) Generate the first trial segment of the slip surface by testing all possibly trial segments that connect all state points in the initial stage to all state points in the second stage.

(5) Calculate the values of the optimal function obtained at all state points in the second stage using Eq. [6], [7] and the assumed F_s. The number of the optimal functions to be calculated at one state point in the second stage is equal to the number of state points located in the initial stage.

(6) Determine the minimum value of the optimal function at each state point in the second stage. The corresponding state point in the previous stage (i.e., the initial stage for the first segment) is identified.

(7) Proceed to the next stage with the same routine until the final stage is reached.

(8) Compare the values of the optimal functions obtained at all state points in the final stage and determine the state point at which the value of the '*optimal function*' is minimum. The determined state point will be the first '*optimal point*' of the '*optimal path*'.

(9) Trace back to the previous stage to find out the corresponding state point with the first '*optimal point*'. This corresponding state point will be the second '*optimal point*' of the '*optimal path*'.

(10) Keep tracing back to the initial stage to determine the whole '*optimal path*'.

(11) Evaluate the actual F_s of this '*optimal path*'. A new value of F_s is assumed based on the values of the first and the actual F_s.

(12) Repeat the procedure until the difference between the assumed and the actual F_s is within the convergence criterion, δ, that is defined as an input value. The '*optimal path*' that corresponds to the final value of F_s will be the critical slip surface.

1.5 *Application*

Three example problems including homogeneous as well as non-homogeneous slopes were solved using the present procedure. Soil properties of three example problems are presented in Table. 1.

Table 1. Soil properties of three example problems.

Problem No.	Layer	Soil properties			
		γ (kN/m^3)	c' (*kPa*)	ϕ'	ϕ^b
1	1	18	20	10	5
2	1	18	10	25	0
	2	15	5	20	0
3	1	15	20	30	0
	2	18	0	10	0
	3	20	100	30	0

Numerous limit equilibrium methods of slices were used to solve these problems in order to compare their solutions with that of the present procedure. Compared methods also include the enhanced method, which was introduced by Kulhawy (1969). Solutions of all example problems using various methods are shown in Fig. 3, 4 and 5.

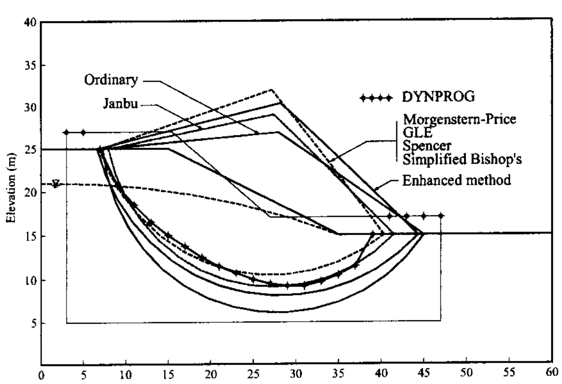

Figure 3. Solution of example 1 obtained by various methods.

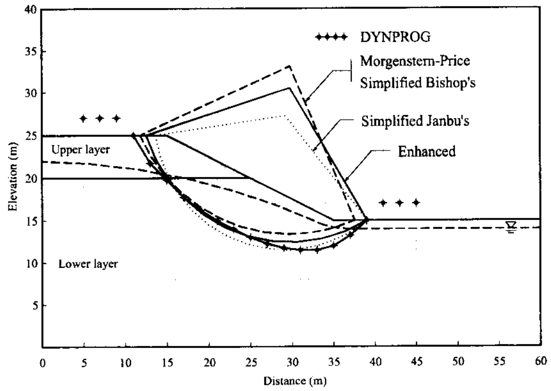

Figure 4. Solution of example 2 obtained by various methods.

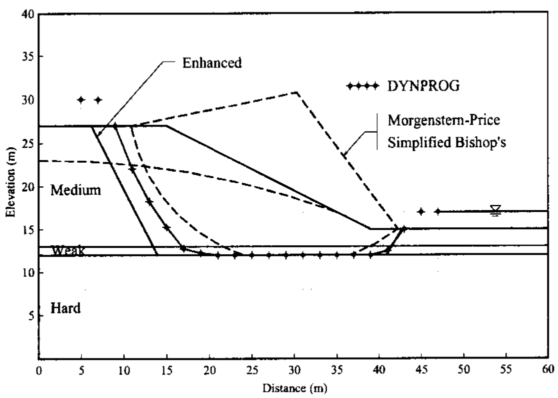

Figure 5. Solution of example 3 obtained by various methods.

CONCLUSIONS

The DP technique combined with finite element method was shown to be applicable for solving slope stability problems. From the above example problems, following conclusions can be made regarding the advancements of the present method:

i) The critical slip surface can be irregular in shape and be automatically determined without using any assumption regarding the location of the potential slip surface.

ii) Using the present method, the mode of failure that used to be anticipated by an analyst is now part of the solution. In other words, there is no assumption required regarding the shape or the location of the critical slip surface except the assumption that the critical slip surface is an assemblage of linear segments.

iii) More sophisticated stress-strain behavior of the soil such as non-linearity can be modeled using finite element method. In other words, the effects of stress history and Poisson's ratio on the stability of the slope can be investigated. These effects were obviously unknown when conventional limit equilibrium methods are used.

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