Are We Approaching The Limits of Limit Equilibrium Analyses?

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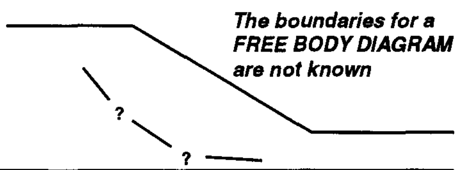
Introduction

- Limit Equilibrium methods of slices have been good for the geotechnical engineering profession since the methods have produced financial benefit
- Engineers are often surprised at the results they are able to obtain from Limit Equilibrium methods

So Why Change?

There are Fundamental Limitations with Limit Equilibrium Methods of Slices



- The SHAPE for the slip surface must be assumed
- The LOCATION of the critical slip surface must be found by TRIAL and ERROR

SHAPE and LOCATION are driving the need for a paradigm shift

Objective of this Presentation:

- To show the gradual change that is emerging in the way that slope stability analyses can be done
- To illustrate the benefits associated with improved procedures for the assessment of stresses in a slope

Outline of Presentation

- Provide a brief Summary of common Limit Equilibrium methods along with their limitations (2-D & 3-D)
- Take the FIRST step forward through use of an independent stress analysis
- Take the SECOND step forward through use of Optimization Techniques

Is a Limit Equilibrium Analysis an Upper Bound or Lower Bound Solution?

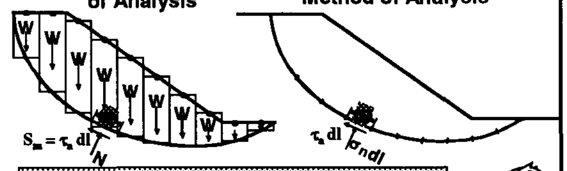
- Limit Equilibrium Methods primarily satisfy requirements of an upper bound type of solution

Reason: the shape of slip surface is selected by the analyst, and thereby a displacement boundary condition is imposed

Limit Formulations from Plasticity

- ❑ **A Lower Bound solution satisfies:**
  - ❑**Equilibrium equation**
  - ❑**Stress boundary conditions**
  - ❑**Failure criterion**
- ❑ **An Upper Bound solution satisfies:**
  - ❑**Velocity (displacement) boundary conditions**
  - ❑**Compatibility conditions**
  - ❑**Failure criterion**

Limit Equilibrium and Finite Element Based Methods of Analyses



QUESTION BEFORE US: How can the Normal Stress at the base of a slice be most accurately computed?

Consider the Free Body Diagram used to calculate the Normal Stress?

Assumption for all Limit Equilibrium Analysis

- **Soils behave as Mohr-Coulomb materials (i.e., soil has friction, ϕ', and cohesion, c')**
- **Factor of safety, F_s, for the cohesive component is equal to the factor of safety for the frictional component**

$$\frac{c'\beta}{F_s} + \frac{[(\sigma_n - u)\tan\phi']\beta}{F_s} = S_m$$

- **The factor of safety is the same for all slices**

Summary of Available Equations Associated with a Limit Equilibrium Analysis

• Equations (knowns):	Quantity
– Moment equilibrium	n
– Vertical force equilibrium	n
– Horizontal force equilibrium	n
– Mohr-Coulomb failure criterion	n
	4n

Summary of Unknowns Associated with a Limit Equilibrium Analysis

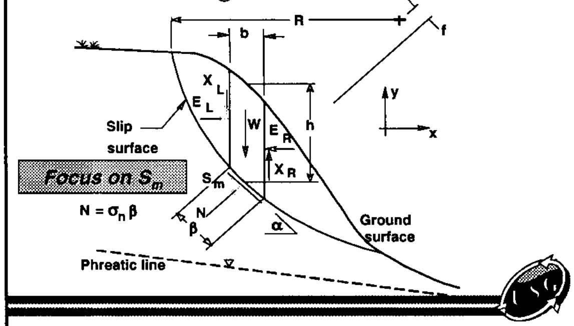
• Unknowns:	Quantity
– Total normal force at base of slice	n
– Shear force at the base of slice, S_m	n
– Interslice normal force, E	n-1
– Interslice shear force, X	n-1
– Point of application of interslice force, E	n-1
– Point of application of normal force	n
– Factor of safety, F_s	1
	6n-2

Definition of Factor of Safety, F_s

❑ ***"That factor by which the shear strength parameters must be reduced to bring the sliding mass into a state of limit equilibrium along a given slip surface"***

$$\frac{c'}{F_s} \quad and \quad \frac{\tan\phi'}{F_s}$$

Forces Acting on Each Slice



Mobilized Shear Force, S_m for Saturated-Unsaturated Soils

$$S_m = \frac{c'\beta}{F_s} + \frac{[(\sigma_n - u_a)\tan\phi']\beta}{F_s} + \frac{[(u_a - u_w)\tan\phi^b]\beta}{F_s}$$

φ^b = *Friction angle with respect to matric suction*

u_a = *Pore-air pressure*

u_w = *Pore-water pressure*

Only new variable required for solving saturated-unsaturated soils problems is the shear force mobilized

- **Moment equilibrium, F_m:**

$$F_m = \frac{\Sigma\left\{c'\beta R + \left(N - u_w\beta\frac{\tan\phi^b}{\tan\phi'}\right)R\tan\phi'\right\}}{\Sigma Wx - \Sigma Nf}$$

- **Force equilibrium, F_f:**

$$F_f = \frac{\sum\left\{c'\beta\cos\alpha + \left(N - u_w\beta\frac{\tan\phi^b}{\tan\phi'}\right)\tan\phi'\cos\alpha\right\}}{\sum N\sin\alpha}$$

Pore-air pressures are assumed to be zero gauge

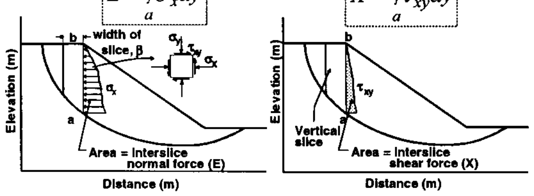
- **Normal force at base of slice:**

$$N = \frac{W - (X_R - X_L) - \frac{c'\beta\sin\alpha}{F} + u_w\frac{\beta\sin\alpha\tan\phi^b}{F}}{\cos\alpha + \frac{\sin\alpha\tan\phi'}{F}}$$

- **Limit Equilibrium methods differ in terms of how (X_R-X_L) is computed and overall statics satisfied**
- **Reality: The Limit Equilibrium problem is indeterminate:**
  - **Can apply an assumption (Historical solution)**
  - **Can utilize additional physics (Future solution)**

Stresses on the Boundary Between Slices

$$E = \int_a^b \sigma_x dy \qquad X = \int_a^b \tau_{xy} dy$$

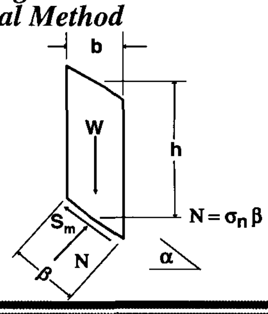


Morgenstern & Price, 1965

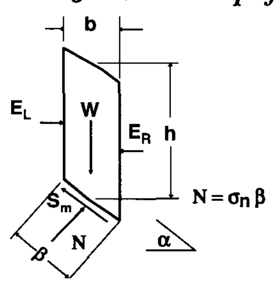
Summary of Limit Equilibrium Methods and Assumptions

Method	Equilibrium Satisfied	Assumptions
Ordinary	Moment, ⊥ to base	*E* and *X* = 0
Bishop's Simplified	Vertical, Moment	*E* is horizontal, *X* = 0
Janbu's Simplified	Vertical, Horizontal	*E* is horizontal, *X* = 0, empirical correction factor, f_o, accounts for interslice shear forces
Janbu's Generalized	Vertical, Horizontal	*E* is located by an assumed line of thrust
Spencer	Vertical, Horizontal, Moment	Resultant of *E* and *X* are of constant slope

Forces Acting on One Slice in Ordinary or Conventional Method



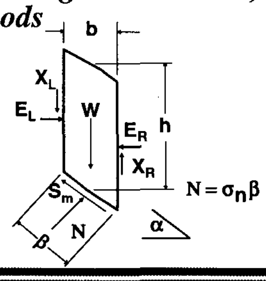
Forces Acting on One Slice in Bishop's Simplified and Janbu's Simplified Methods

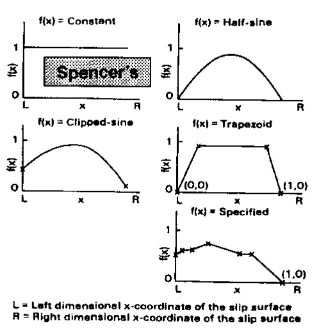


Summary of Limit Equilibrium Methods and Assumptions

Method	Equilibrium Satisfied	Assumptions
Morgenstern-Price, GLE	Vertical Horizontal Moment	Direction of *E* and *X* is defined by an arbitrary function. Percent of the function required to satisfy moment and force equilibrium is called λ
Corps of Engineers	Vertical Horizontal	Direction of *X* and *E* is parallel to the ground
Lowe and Karafiath	Vertical Horizontal	Direction of *X* and *E* is the average of the ground surface slope and the slope at the base of a slice

Forces Acting on One Slice in Spencer's, Morgenstern-Price, and GLE Methods

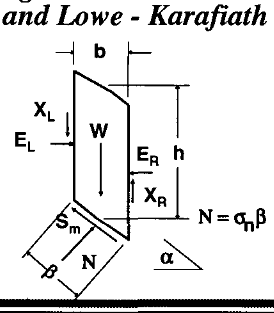


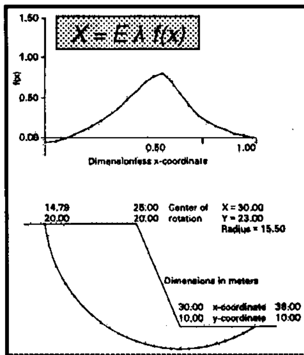


Various Interslice Force Functions

Proposed by Morgenstern & Price (1965)

Forces Acting on One Slice in Corps of Engineers, and Lowe - Karafiath Methods



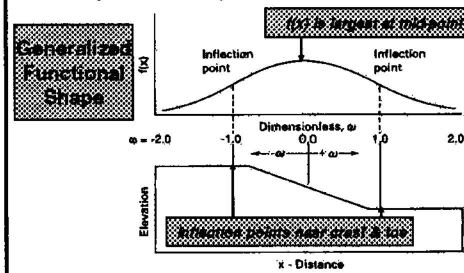


Wilson and Fredlund (1983)

Used a finite element stress analysis (with gravity switched on) to determine a shape for the Interslice Force Function

Interslice Force Function for a Deep-seated Slip Surface Through a 1:2 Slope

Definition of Dimensionless Distance



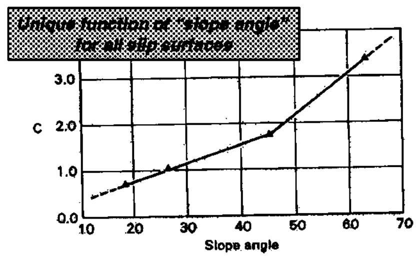
$X = E\lambda f(x)$

Wilson and Fredlund, 1983

$$f(x) = Ke^{-\left(C^n \omega^n\right)/2}$$

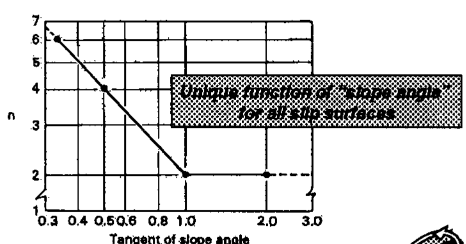
Dimensionless Distance

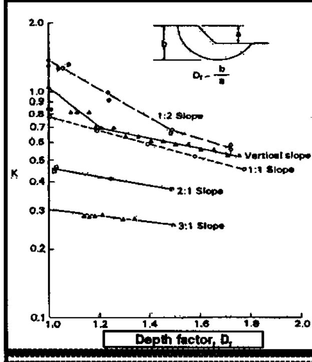
where:
K = magnitude of function at mid-slope
e = base of natural log
C =variable to define inflection point
n = variable to define steepness
w = dimensionless x-position



"C" coefficient versus slope angle

"n" coefficient versus tangent of slope angle

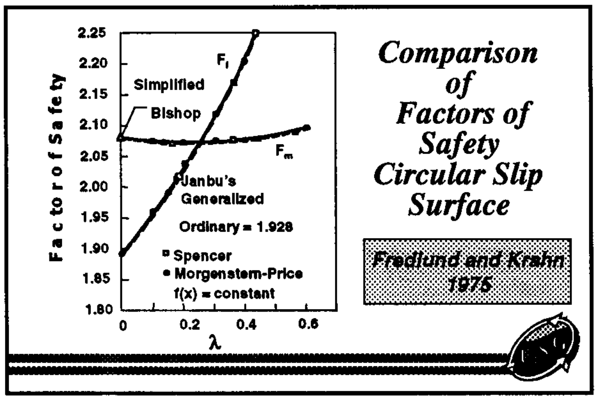


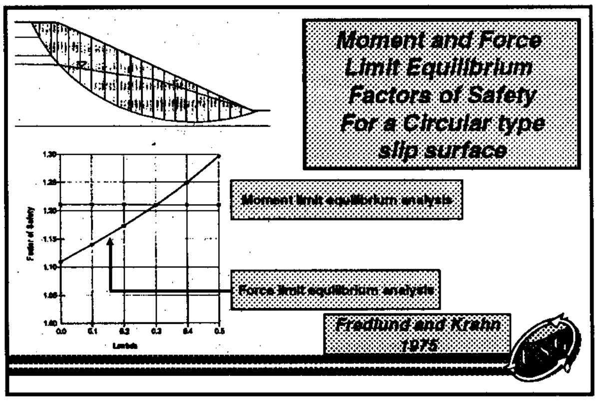


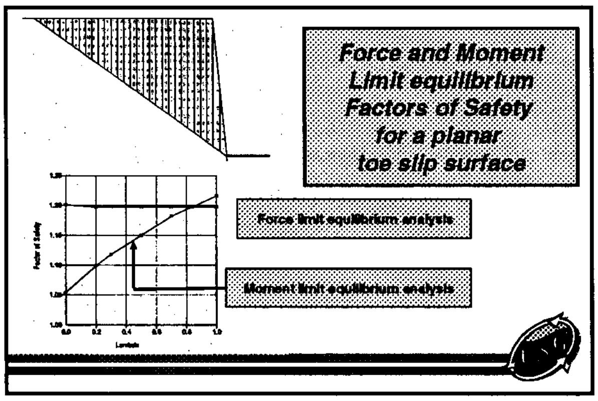
$X = E\lambda f(x)$

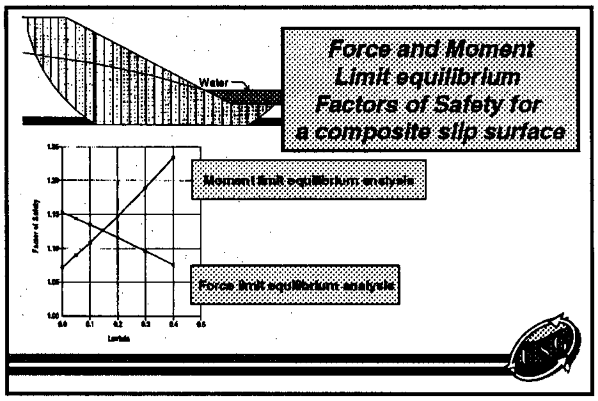
Interslice Side Force Ratio, K, at Mid-slope versus Depth Factor, D_f

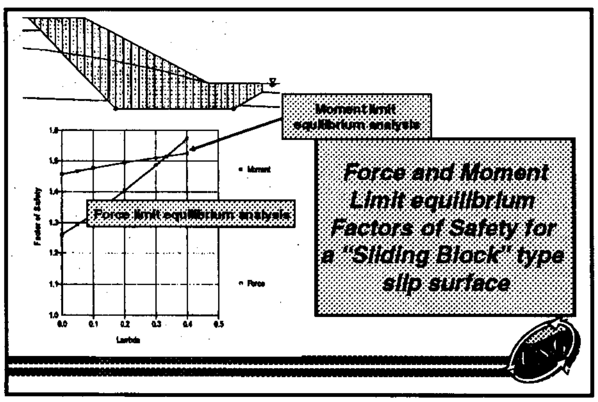
Wilson, 1983





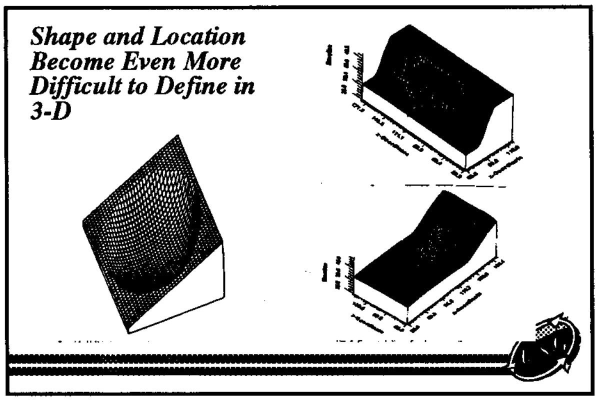


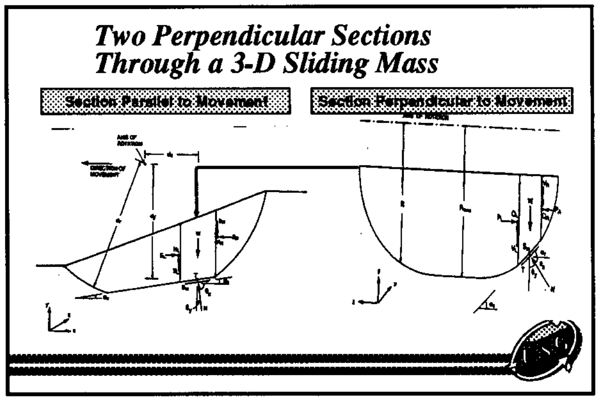


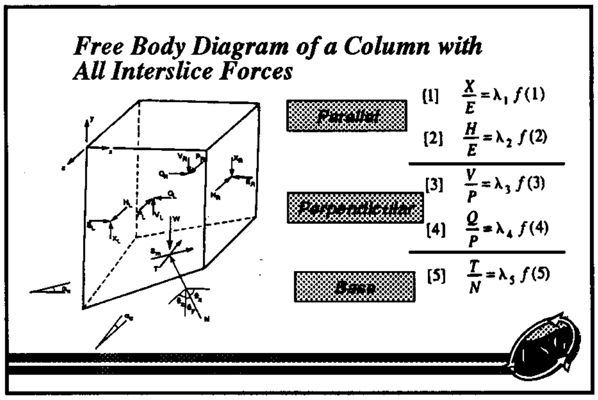


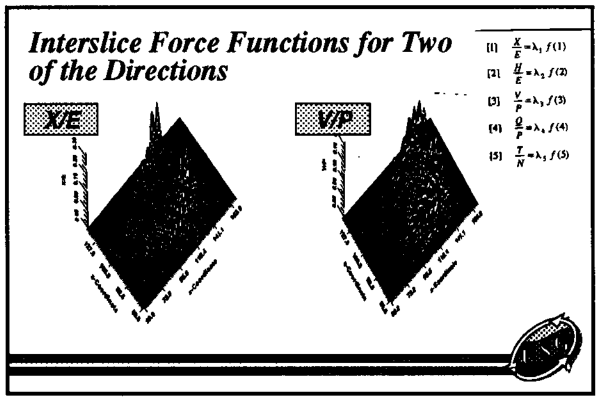
Extensions of Methods of Slices to Three-dimensional Methods of Columns

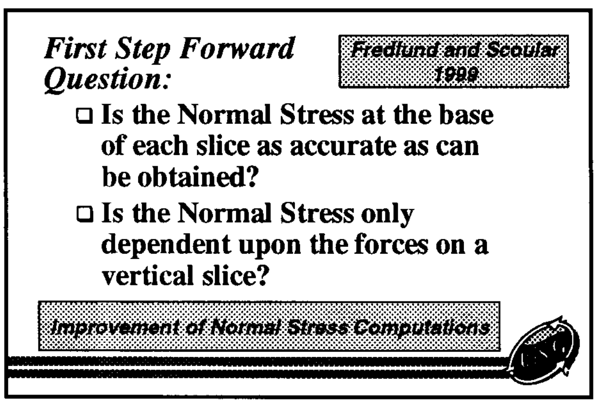
- **Hovland (1977) – 3-D of Ordinary**
- **Chen and Chameau (1982) – 3-D of Spencer**
- **Cavounidis (1987) – 3-D F_s > 2-D F_s**
- **Hungr (1987) – 3-D of Bishop Simplified**
- **Lam and Fredlund (1993) – 3-D with *f(x)* on all 3 planes; 3-D of GLE**

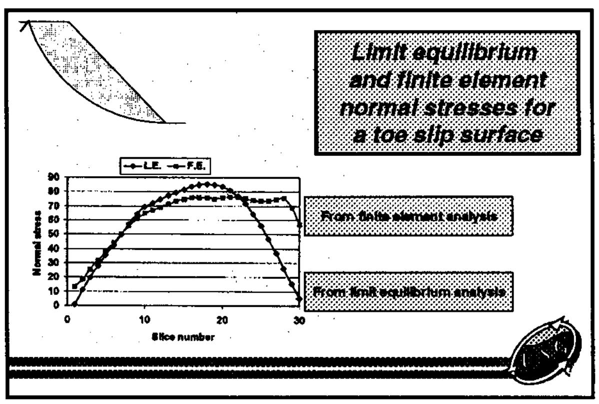

Shape and Location Become Even More Difficult to Define in 3-D


Two Perpendicular Sections Through a 3-D Sliding Mass
Section Parallel to Movement
Section Perpendicular to Movement

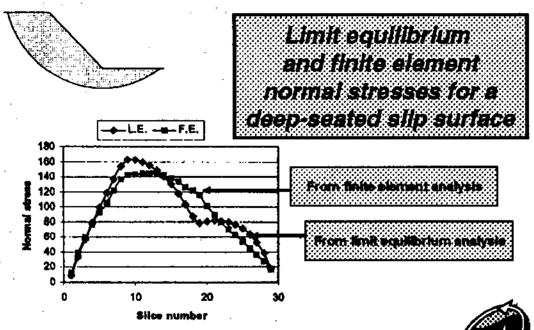

Free Body Diagram of a Column with All Interslice Forces
Parallel
Perpendicular
Base
[1] $\frac{X}{E} = \lambda_1 f(1)$
[2] $\frac{H}{E} = \lambda_2 f(2)$
[3] $\frac{V}{P} = \lambda_3 f(3)$
[4] $\frac{Q}{P} = \lambda_4 f(4)$
[5] $\frac{T}{N} = \lambda_5 f(5)$


Interslice Force Functions for Two of the Directions
X/E
V/P
[1] $\frac{X}{E} = \lambda_1 f(1)$
[2] $\frac{H}{E} = \lambda_2 f(2)$
[3] $\frac{V}{P} = \lambda_3 f(3)$
[4] $\frac{Q}{P} = \lambda_4 f(4)$
[5] $\frac{T}{N} = \lambda_5 f(5)$

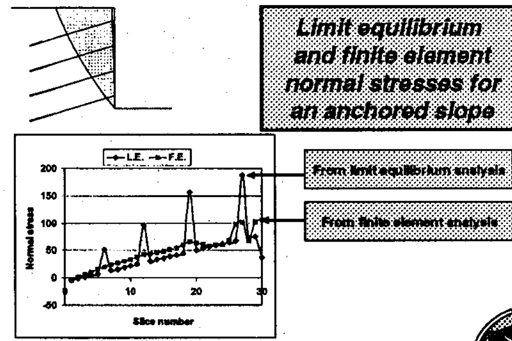

First Step Forward Question:
Fredlund and Scoular 1999
❑ Is the Normal Stress at the base of each slice as accurate as can be obtained?
❑ Is the Normal Stress only dependent upon the forces on a vertical slice?
Improvement of Normal Stress Computations


Limit equilibrium and finite element normal stresses for a toe slip surface
L.E.
F.E.
From finite element analysis
From limit equilibrium analysis
Normal stress
Slice number
0
10
20
30
40
50
60
70
80
90

Limit equilibrium and finite element normal stresses for a deep-seated slip surface



Limit equilibrium and finite element normal stresses for an anchored slope



Using Limit Equilibrium Concepts in a Finite Element Slope Stability Analysis

Objective:

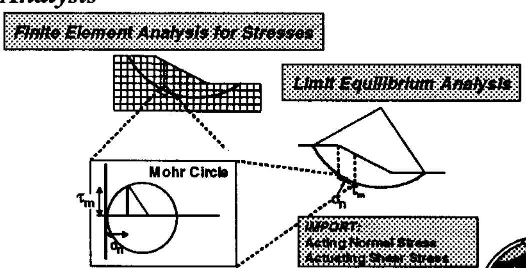
- To illustrate procedures for combining a finite element stress analysis with concepts of limiting equilibrium. (i.e., finite element method of slope stability analysis)
- To compare results of a finite element slope stability analysis and conventional limit equilibrium methods

Hypothesis

- *The complete stress state from a finite element analysis can be "imported" into a limit equilibrium framework where the normal stress and the actuating shear stress are computed for any selected slip surface*

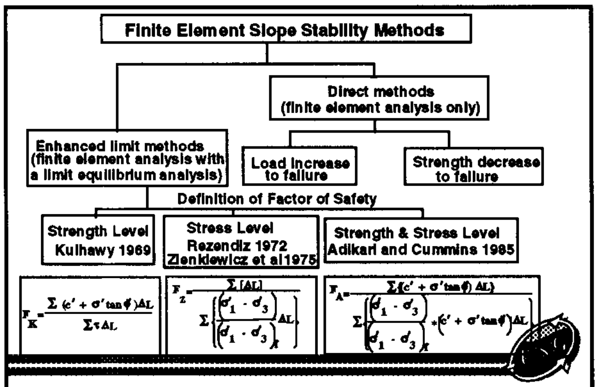
Assumption: The stresses computed from "switching-on" gravity are more reasonable than the stresses computed on a vertical slice

Manner of "Importing" Stresses from a Finite Element Analysis into a Limit Equilibrium Analysis



Background to Using Other Stress Analyses

- **Bishop (1952) - stresses from Limit Equilibrium methods do not agree with actual soil stresses**
- **Clough and Woodward (1967) - "meaningful stability analysis can be made only if the stress distribution within the structure can be predicted reliably"**
- **Kulhawy (1969) - used normal and shear stresses from an elastic analysis to compute factor of safety "Enhanced Limit Strength Method"**



Differences and Similarities Between the Finite Element Slope Stability and Conventional Limit Equilibrium

- **Differences**
  - **Determinate**
  - **Factor of safety equation is linear**
- **Similarities**
  - **Still necessary to assume the shape of the slip surface and search by trial and error to locate the critical slip surface**

Why hasn't Finite Element Slope Stability Method been extensively used?

- **Difficulties and perceptions related to the stress analysis**
- **Inability to transfer large amounts of data and find needed information**

Now: Microcomputer have dramatically changed our ability to combine Finite Element and Limit Equilibrium analyses

Definition of Factor of Safety

- **Kulhawy (1969)**

$$F_{FEM} = \frac{\Sigma S_r}{\Sigma S_m}$$

Actuating Shear

- **where:**
  - **S_r = resisting shear strength or**

$$S_r = \{c' + (\sigma_n - u_w) \tan \phi'\} \beta$$

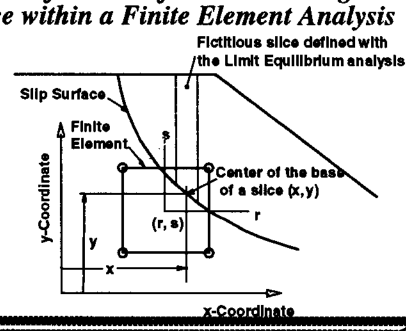
  - **S_m = mobilized shear force**

Normal Stress

Analysis Study Undertaken by Fredlund and Scoular (1999)

- **Adopted the Kulhawy (1969) procedure**
- **Used Sigma/W and Slope/W**
- **Poisson's ratio range = 0.33 to 0.48**
- **Elastic modulus, E = 20,000 to 200,000 kPa**
- **Cohesion, c' = 10 to 40 kPa**
- **Friction, ϕ' = 10 to 30 degrees**
- **Compared conventional Limit Equilibrium results with Finite Element slope stability results**

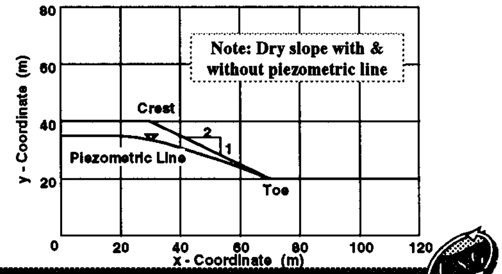
Location of Center of a Section along the Slip Surface within a Finite Element Analysis



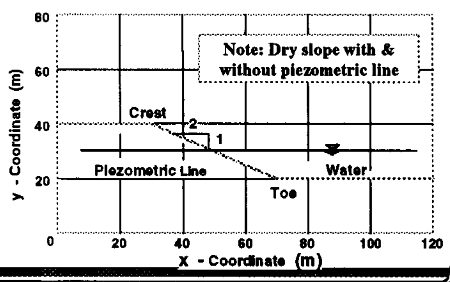
Presentation of Finite Element Slope Stability Results

- **Conditions Analyzed:**
  - ❑ **Dry slope**
  - ❑ **Piezometric line at 3/4 height, exiting at toe**
  - ❑ **Dry slope, partially submerged**
  - ❑ **Piezometric line at 1/2 height and submerged to mid-height**

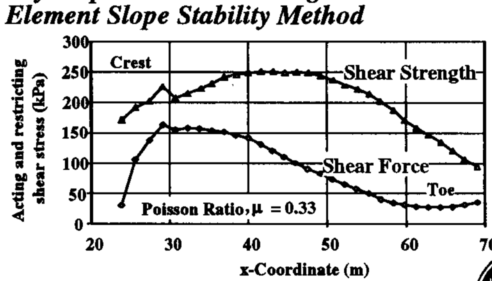
Selected 2:1 Free-Standing Slope with a Piezometric Line Exiting at the Toe of the Slope



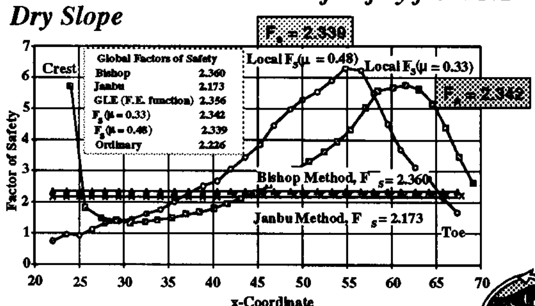
Selected 2:1 Partially Submerged Slope with a Horizontal Piezometric Line at Mid-Slope



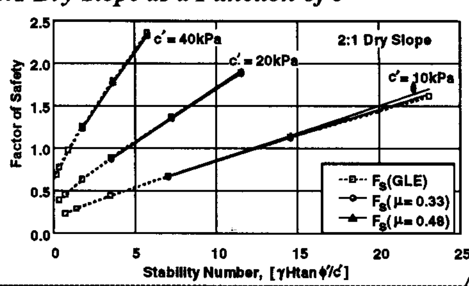
Shear Strength and Shear Force for a 2:1 Dry Slope Calculated Using the Finite Element Slope Stability Method



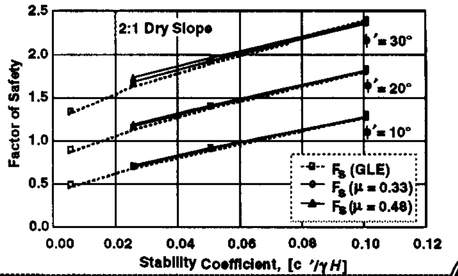
Local and Global Factors of Safety for a 2:1 Dry Slope



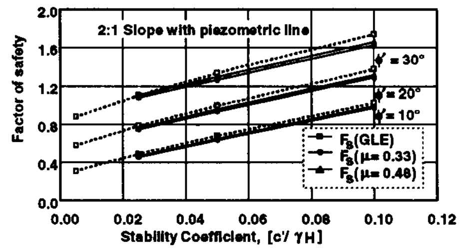
Factors of Safety Versus Stability Number for a 2:1 Dry Slope as a Function of c'



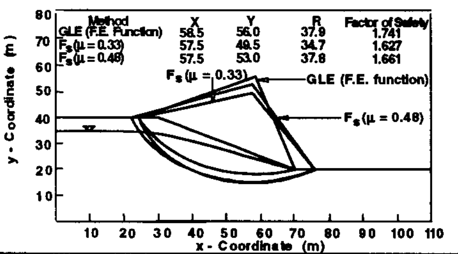
Factor of Safety Versus Stability Coefficient for a 2:1 Dry Slope as a Function of ϕ'



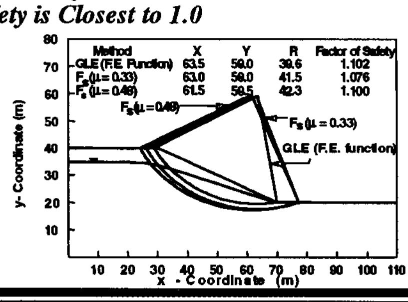
Factor of Safety Versus Stability Coefficient as a Function of ϕ' for 2:1 Slope with a Piezometric Line



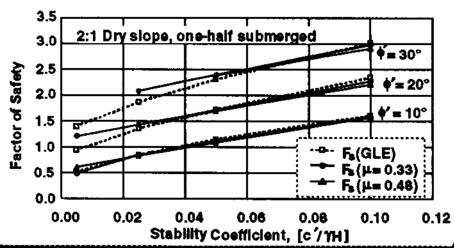
Location of the Critical Slip Surface for a Slope with a Piezometric Line with Soil Properties of $c' = 40$ kPa and $\phi' = 30°$



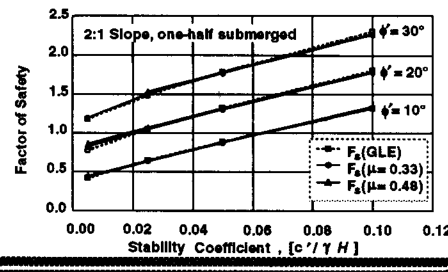
Location of the Critical Slip Surface for a Slope with a Piezometric Line where the Factor of Safety is Closest to 1.0



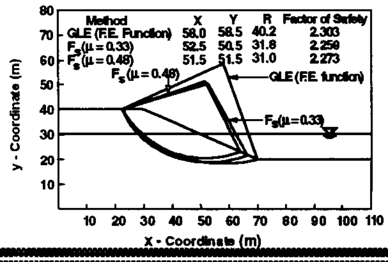
Factor of Safety Versus Stability Coefficient as a Function of ϕ' for 2:1 Dry Slope, 1/2 Submerged



Factor of Safety Versus Stability Coefficient as a Function of ϕ' for 2:1 Slope Half Submerged with Piezometric Line



Location of the Critical Slip Surface for a Half Submerged Slope where the Soil Properties are $c' = 40$ kPa and $\phi' = 30°$



Conclusions from Step 1 Forward

- **Normal and Actuating Shear stresses from a finite element analysis appears to provide a more accurate representation of the stress state in a slope**
- **The Enhanced Limit method by Kulhawy (1969) appears to open the way to simulate more complex slope stability problems**
- **Enhanced Limit methods can readily be used in routine engineering practice**

How do the Results from Enhanced Limit Methods Compare to Limit Equilibrium Methods?

- **Global factors of safety appear to be essentially the same for most simple slopes**
- **Selection of Poisson's ratio has some effect on the Enhanced factor of safety**
- **Factors of Safety appear to differ slightly at:**
  - **Low cohesion values**
  - **High angles of internal friction**

Local Factors of Safety can also be computed by the Enhanced Method

Second Step Forward Question:

Ha and Fredlund 2002

- **Is it possible for the computer to determine the Shape of the critical slip surface?**
- **Is it possible for the computer to determine the Location of the critical slip surface?**

Improvement on Shape and Location

Hypothesis

- ***Optimization Techniques (i.e., Dynamic Programming) can be used to find the pathway which minimizes a function of the shear strength available to the actuating shear stress within a soil mass***

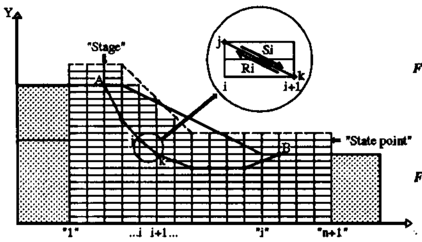
Assumption: The stresses computed from "switching-on" gravity can be used to represent the stress state in the soil mass

Slope Stability Analysis Using Dynamic Programming Combined with a Finite Element Stress Analysis

- **Dynamic Programming (DP) optimization techniques for slope stability analysis (Spencer's Method) was introduced by Baker (1980)**
- **Yamagami & Ueta (1988) and Zou et al. (1995) improved on the Baker (1980) solution by coupling Dynamic Programming with a Finite Element stress analysis**

Definition of Factor of Safety

$F_s = \Sigma$(Shear Strength) / Σ (Actuating Shear Stress)



Smooth curve

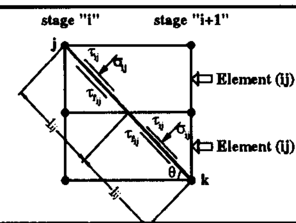
$$F_s = \frac{\int_A^B \tau_f \, dL}{\int_A^B \tau \, dL} \quad (1)$$

Discrete form

$$F_s = \frac{\sum_{i=1}^{n} \tau_{f_i} \Delta L_i}{\sum_{i=1}^{n} \tau_i \Delta L_i} \quad (2)$$

Definition of "Return Function"; G

$F_s = \Sigma$(Shear Strength) / Σ (Actuating Shear Stress) Difficult to minimize !



$$G = \sum_{i=1}^{n} (\tau_{fi} - F_s \tau_i) \Delta L_i$$

$$G = \int_A^B (\tau_f - F_s \tau) dL$$

$$G = \sum_{i=1}^{n} (R_i - F_s S_i)$$

R = Resisting Shear Strength; S = Actuating Shear Stress

Actuating Shear Forces and Shear Resistance

$$S_i = \tau_i \Delta L_i = \sum_{ij=1}^{ne} S_{ij} = \sum_{ij=1}^{ne} \tau_{ij} l_{ij}$$

S = Actuating Shear Stress

$$R_i = \tau_{f_i} \Delta L_i = \sum_{ij=1}^{ne} R_{ij} = \sum_{ij=1}^{ne} \tau_{f_{ij}} l_{ij}$$

R = Resisting Shear Strength

$$R_i = \sum_{ij=1}^{ne} \{c'_{ij} + (\sigma_{ij} - u_a) \tan\phi'_{ij} + (u_a - u_w) \tan\phi^b_{ij}\} l_{ij}$$

Definition of "Optimal Function": Minimum Value of "Return Function"

$$G_{\min} = \min G = \min \sum_{i=1}^{n} (R_i - F_s S_i)$$

G = Return Function

Introduce an "optimal function", $H_i(j)$

$$H_{i+1}(k) = H_i(j) + G_i(\overline{j,k})$$

H = Optimal Function

where:

$H_{i+1}(k)$ *= the optimal function obtained at point {k} of stage [i+1],*

$H_i(j)$ *= the optimal function obtained at point {j} in stage [i], and*

$G_i(\overline{j,k})$ *= the return function calculated when passing from the state point {j} in stage [i] to the state point {k} in stage [i+1].*

Boundary Conditions of "Optimal Function"

At the initial stage, (i=1) :

$$H_1(j) = 0 \qquad j = 1 \ldots NP_1$$

At the final stage, (i = n+1) :

H = Optimal Function

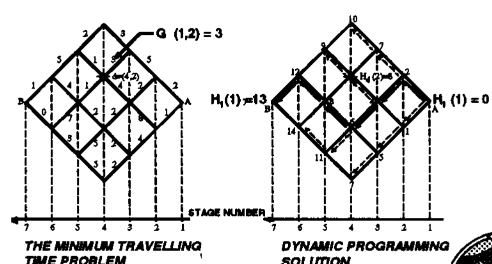
$$H_{n+1}(k) = H_n(j) + G_n(\overline{j,k})$$

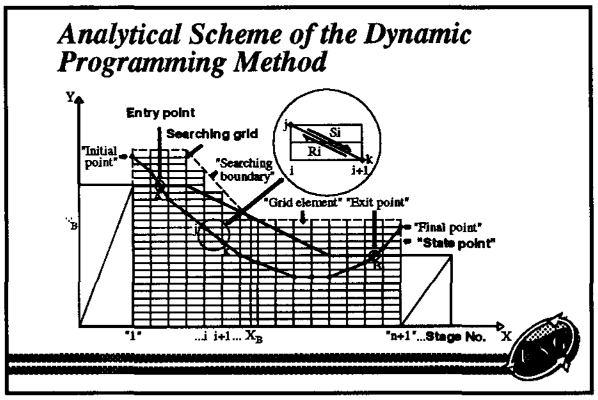
$$H_{n+1}(k) = G_m = \sum_{i=1}^{n} (R_i - F_s S_i) \qquad k = 1 \ldots NP_{n+1}$$

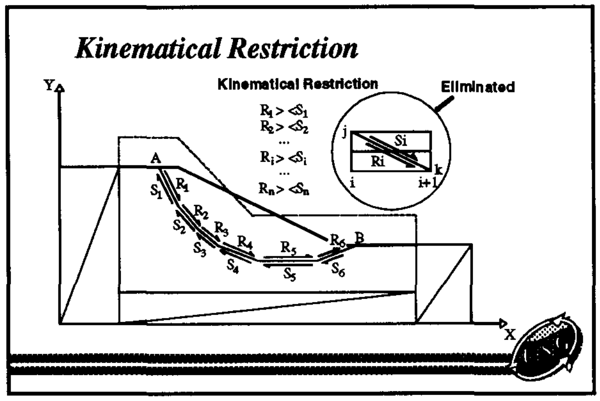
where:

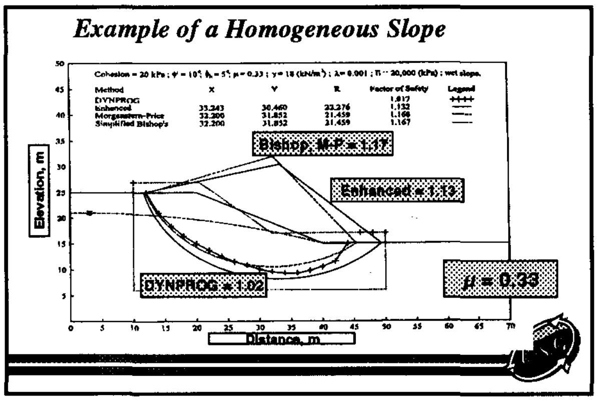
NP_{n+1} *= the number of state points in the final stage*

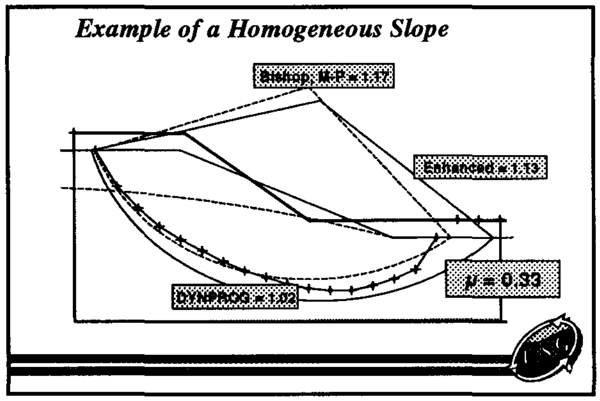
The Minimum (or Optimal) Travelling Time Problem




Analytical Scheme of the Dynamic Programming Method
Entry point
Searching grid
"Initial point"
"Searching boundary"
"Grid element" "Exit point"
"Final point"
"State point"
Si
Ri
"1" ...i i+1... X
"n+1"...Stage No.


Kinematical Restriction
Kinematical Restriction
Eliminated
Si
Ri


Example of a Homogeneous Slope
Bishop; M-P = 1.17
Enhanced = 1.13
DYNPROG = 1.02
μ = 0.33
Elevation, m
Distance, m


Example of a Homogeneous Slope
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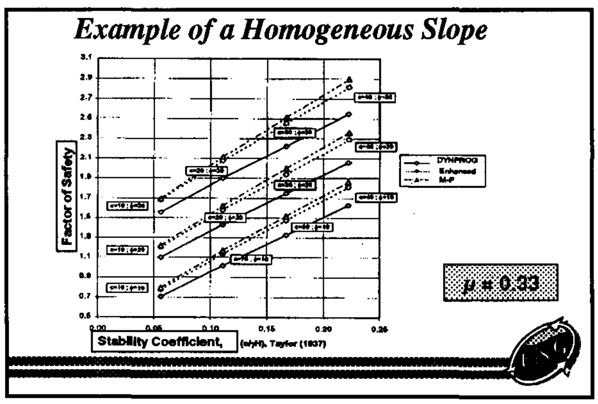
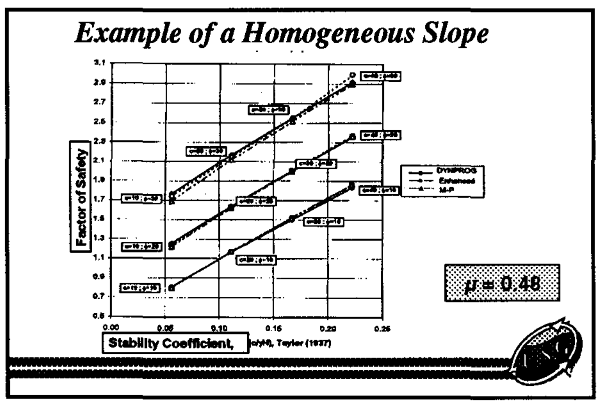
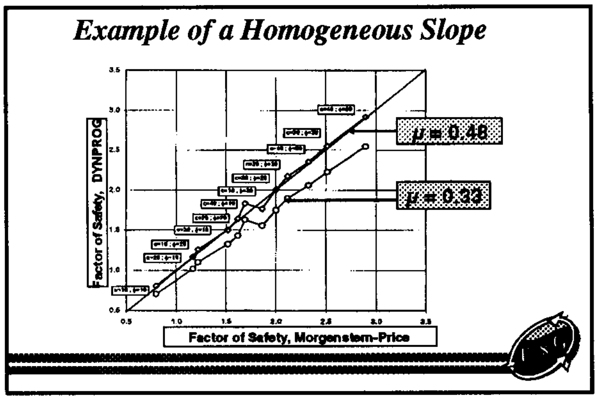
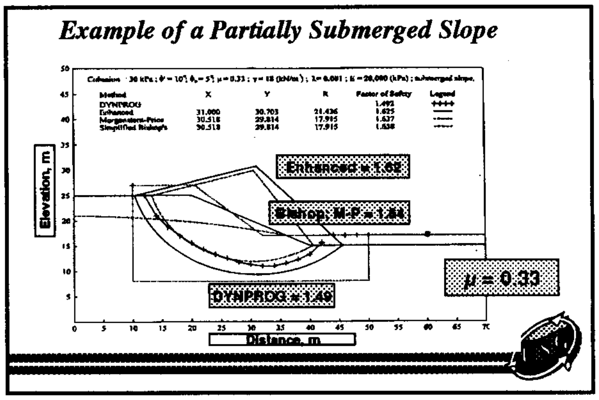
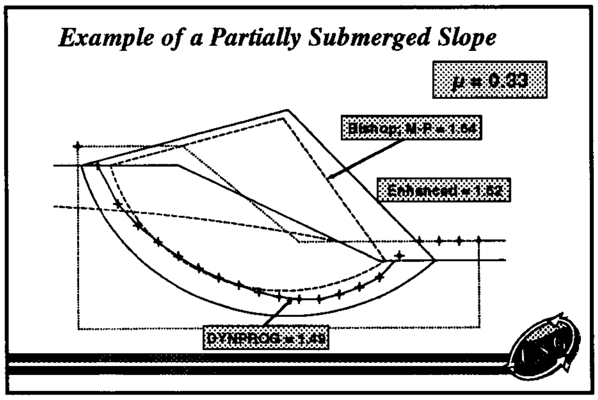
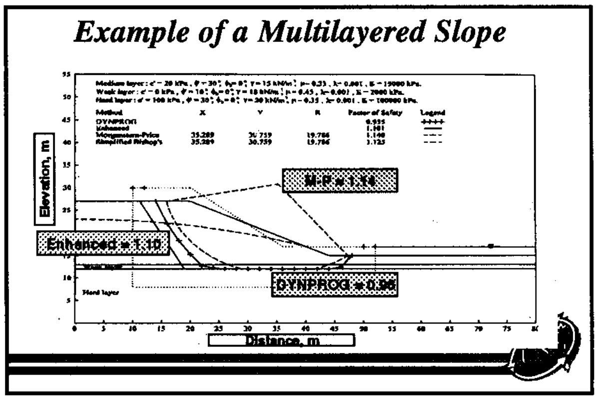

Example of a Homogeneous Slope
Factor of Safety
Stability Coefficient,
μ = 0.33

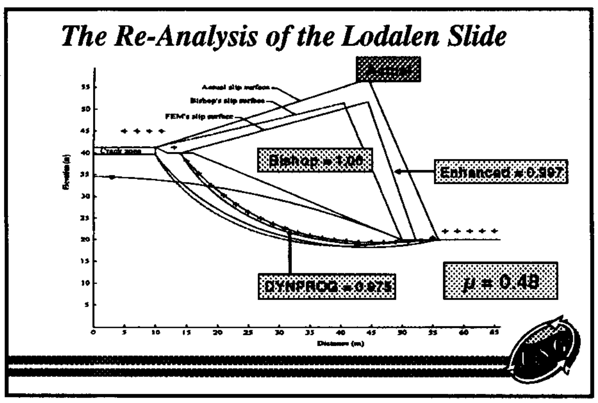

Example of a Homogeneous Slope
Factor of Safety
Stability Coefficient,
μ = 0.48

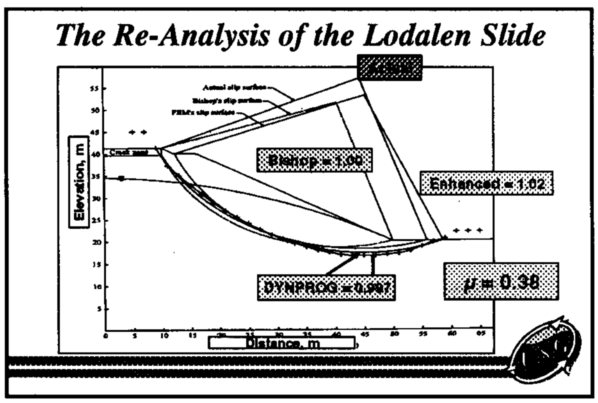

Example of a Homogeneous Slope
Factor of Safety, DYNPROG
Factor of Safety, Morgenstern-Price
μ = 0.48
μ = 0.33


Example of a Partially Submerged Slope
Elevation, m
Distance, m
Enhanced = 1.62
Bishop; M-P = 1.64
DYNPROG = 1.49
μ = 0.33

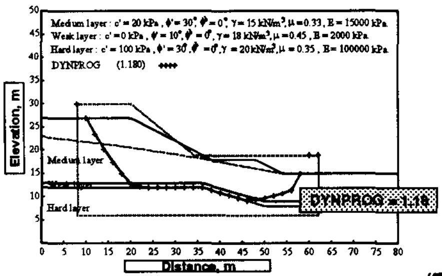

Example of a Partially Submerged Slope
μ = 0.33
Bishop; M-P = 1.64
Enhanced = 1.62
DYNPROG = 1.49


Example of a Multilayered Slope
Elevation, m
Distance, m
M-P = 1.14
Enhanced = 1.10
DYNPROG = 0.96

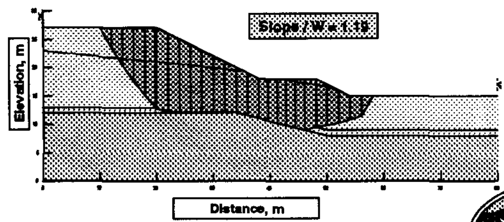

The Re-Analysis of the Lodalen Slide
Actual
Bishop = 1.00
Enhanced = 0.997
DYNPROG = 0.875
μ = 0.48


The Re-Analysis of the Lodalen Slide
Actual
Elevation, m
Distance, m
Bishop = 1.00
Enhanced = 1.02
DYNPROG = 0.987
μ = 0.38

Example Problem Involving the Search for a Convex Critical Slip Surface Along a Weak Clay Layer



Solution of the Concave Slip Surface Problem Using Slope/W Once the Critical Slip Surface has been Defined



Conclusions from Step 2 Forward

- **The Shape of the critical slip surface can be made part of the solution**
- **The critical slip surface can be irregular in shape but must be kinematically admissible**
- **No assumptions is required regarding the Location of the critical slip surface which is defined as an assemblage of linear segments**
- **Force and moment equilibrium equations are satisfied through the stress analysis**

Recommendations for the Future

- ***The normal and shear stresses should be studied using more sophisticated stress-strain nonlinear and elasto-plastic models including Poisson's ratio effects***
- ***Study of "true" 3-dimensional modelling of slopes***
- ***Further study of case histories would be valuable***
- ***Dynamic Programming should be applied to Lateral Earth Pressure and Bearing Capacity problems***

Delwyn G. Fredlund

INBOX: ASCE(HK) Seminar on Slope Stability Analysis (117 of 122)

Date: Tue, 11 Feb 2003 10:49:44 +0800
From: GE/lpmwr1 <gelpmwr1_mwd@ced.gov.hk>
To: Dr Fredlund <Fredlund@ust.hk>, "Dr. Y M Cheng" <ceymchen@polyu.edu.hk>
Cc: Dr L M Zhang <cezhangl@ust.hk>, "Dr Mark (@net) Chan" <hcmarkchan@netvigator.com>
Subject: ASCE(HK) Seminar on Slope Stability Analysis

Dear Prof. Fredlund and Prof. Cheng,

The last seminar on 18 Jan was obviously very successful.
Dr Limin Zhang of HKUST, MASCE, suggested that another similar seminar can be organized by ASCE.
I have proposed the seminar to the Board of Directors on 10 Feb and the proposal was very much welcomed.
I thus request your agreement to give a presentation on slope stability analysis with the following details:

Date: sometime before end of June, perhaps a Saturday morning?
Venue: HKUST (to be provided free of charge by UST with assistance of Dr Zhang)
Seminar fee: Free
Target audience: geotechnical practitioners, some civil engineering students
Refreshment: subsidized by ASCE (up to $1000)

Apart from the topics presented last time, I consider that common pitfalls and problematic cases would be interesting to the practising community.

Should you agree to give the presentation, please indicate preferred dates, outline of talk, and introduction to speaker. Please reply by 18 Feb.

Looking forward to hear from you.

HAPPY CHINESE NEW YEAR!

Regards,
Mark Chan
President-Elect, ASCE(HK)