A comparative study of constitutive models for unsaturated soils

E.C. Leong, H. Rahardjo
Nanyang Technological University, School of Civil & Environmental Engineering, Singapore
D.G. Fredlund
University of Saskatchewan, Canada

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ABSTRACT: Many geotechnical problems are related to the shear strength and volume change behaviour of a soil. For saturated soils, numerous elasto-plastic constitutive models are available. These models have parameters as few as two but some can have as many as over 20 parameters. Some of these models have migrated into the realm of unsaturated soil mechanics. Extrapolating from the experiences in saturated soils, it is easy to see that constitutive models are dependent on soil types, stress conditions, loading conditions and sample preparation. It may be difficult to find a single unique constitutive model that is applicable for all soil types, stress conditions and loading conditions. Fortuitously, development of constitutive models for unsaturated soils is in its infancy due, in part to the difficulties and long duration of tests for unsaturated soils. It is timely to examine the constitutive models for unsaturated soils and to assess their individual merits and discrepancies. The constitutive relations examined in this paper are for the *q-p-s* space and the *e-p-s* space which are formulated under the framework of elasto-plasticity. The models are examined in the light of published experimental data. Useful features of some models are highlighted which should be further investigated to provide the basis for improving the constitutive models of unsaturated soils.

1 INTRODUCTION

A constitutive equation is a mathematical model which permits the reproduction of the response of a continuum (Desai & Siriwardane 1984). Several types of constitutive relations exist for unsaturated soils (Fredlund & Rahardjo 1993): stress versus volume-mass, stress versus stress and stress gradient versus flow rate. In this paper, the scope will be restricted to the stress versus stress and stress versus volume-mass constitutive relations.

A satisfactory constitutive model should be complete in the sense that the material behaviour is modeled for all stress and strain paths; it should be possible to obtain the model parameters through a reasonable number of tests and the model should be founded upon physical principles of the soil response to applied stress or strain (Prevost & Popescu 1996). There are a vast number of constitutive models for saturated soils and the number of parameters required in each model can range from two to over 20 parameters. The implication of this is that most of these constitutive models are only satisfactory for certain stress and strain paths and may only be applicable to only certain soil types. Some of these constitutive models have found their way into the realm of unsaturated soil mechanics.

To a practitioner, a constitutive model is only justifiable if it can be used to solve engineering problems immediately or if it is a planned step towards providing a future model that can be used eventually for solving engineering problems (Marti 1984). The current interest in unsaturated soils will certainly lead to a proliferation of constitutive models for unsaturated soils. It is perhaps timely to take stock of the current constitutive models that have already been developed for unsaturated soils and highlight the merits and discrepancies for future developments. Constitutive models are formulated using theories of elasticity, hypoelasticity and plasticity. A survey of the literature on constitutive modeling of unsaturated soils shows that a number of the constitutive models are elasto-plastic models and these models are the subject of the present paper.

2 STRESS STATE VARIABLES

The mechanical behaviour of an unsaturated soil is governed by three stress variables: total stress, σ, pore-water pressure, u_w, and pore-air pressure, u_a. These three stress variables can be combined to give two stress state variables for unsaturated soils (Table 1). Any of the three combinations can be used in the formulation of a constitutive model.

Table 1. Possible combinations of stress state variables for an unsaturated soil (after Fredlund & Rahardjo 1993)

Reference Pressure	Stress State Variables
Air, u_a	(σ - u_a) and (u_a - u_w)
Water, u_w	(σ - u_w) and (u_a - u_w)
Total, σ	(σ - u_a) and (σ - u_w)

There have been many attempts to combine the stress variables into an equivalent "effective stress" to saturated soils. The earlier attempts have been listed in Fredlund & Morgenstern (1977). More recently, similar attempts have also been made. Kohgo et al. (1993) suggested the following effective stress:

$$\begin{aligned}&\sigma' = \sigma - u_a - s \quad (s \le s_e)\\&\sigma' = \sigma - u_a - \left(s_e + \frac{s_c - s_e}{s - s_e + a_e}(s - s_e)\right) \quad (s > s_e)\end{aligned} \tag{1}$$

where: $s = (u_a\text{-}u_w)$ = suction, s_e = air-entry value, s_c = critical suction and a_e = material parameter. Khalili & Khabbaz (1998), Loret & Khalili (2000) and Simoni & Schrefler (2001) supported the use of the Bishop's effective stress:

$$\sigma' = (\sigma - u_a) + \chi(u_a - u_w) \tag{2}$$

where χ is related to degree of saturation.

Jommi (2000) had argued for the use of an average skeleton mean stress, $\hat{p}$, given by:

$$\hat{p} = (p - u_a) + S_r s \tag{3}$$

where p is mean stress and S_r is the degree of saturation.

As correctly pointed out by Wheeler & Karube (1996), such proposed definitions of an equivalent "effective stress" will not be able to represent some of the most fundamental behaviours of unsaturated soils. In this paper, only the elasto-plastic models where the two stress state variables, ($\sigma - u_a$) and ($u_a - u_w$), are used will be discussed.

3 CONSTITUTIVE MODELS

The elasto-plastic framework has been used succefully to describe many features of unsaturated soils (Alonso et al. 1990, Kohgo et al. 1993, Wheeler & Sivakumar 1995, Maâtouk et al. 1995, Cui & Delage 1996, Karube 1997, Geiser et al. 2000, Vaunat et al. 2000, Sun et al. 2000, Tang & Graham 2002). The ideas embodied in these models are best presented in the *q-p-s* space (stress versus stress constitutive relation) and in the *e-p-s* space (stress versus volume constitutive relation) and therefore the discussions will be divided into these two constitutive relations.

3.1 *The q-p-s space*

In the critical state model for saturated soils, a yield surface and a failure line are defined in the (*q-p*) space (Fig. 1). The failure line or critical state line for saturated soils is given by:

$$q = M(p - u_w) = Mp' \tag{4}$$

where M is a function of the effective friction angle of the soil.

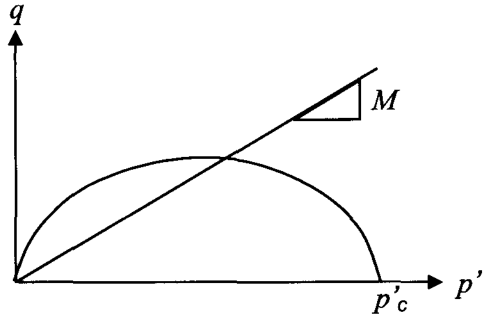


Figure 1. Yield surface and failure line in *q-p* space.

Alonso et al. (1990) had extended Equation 4 for unsaturated soils as:

$$q = M[(p - u_a) + k(u_a - u_w)] \tag{5}$$

where k is a constant. Jommi (2000) suggested a similar equation for the failure line:

$$q = M[(p - u_a) + S_r(u_a - u_w)] = M\hat{p} \tag{6}$$

where $\hat{p}$ is the average soil skeleton mean stress. Following Fredlund et al. (1978) extended Mohr Coulomb model for unsaturated soils, Toll (1990) suggested that the failure line be given by:

$$q = M_a(p - u_a) + M_w(u_a - u_w) \tag{7}$$

where M_a and M_w are functions of degree of saturation and the soil fabric. Wheeler & Sivakumar (1995) suggested that the failure line be represented by:

$$q = M_s(p - u_a) + \mu_s \tag{8}$$

where M_s and μ_s are functions of suction. Maâtouk et al. (1995) suggested the following equation for the failure line:

$$q = M(p - u_a) + [b_1 - b_2(p - u_a)](u_a - u_w) \tag{9}$$

where b_1 and b_2 is dependent on (p-u_a). Karube (1997) used the following for the failure line:

$$q = M[(p - u_a) + \chi s] - [(p - u_a) + \chi_b s]\left(\frac{dv^p}{d\varepsilon}\right) \tag{10}$$

where χ is the relative projected area of water, χ_b is the relative projected area of bulk water, v^p is the plastic volumetric strain and ε is the axial strain. Sun et al. (2000) proposed the following equation for the failure line:

$$q = M_s\left(p + \frac{(u_a - u_w)}{1 + \frac{(u_a - u_w)}{a}}\right) \tag{11}$$

where M_s is a function of suction and a is the limit maximum suction for a soil at high suctions. Tang and Graham (2002) combined $(p\text{-}u_a)$ and $(u_a\text{-}u_w)$ into an equivalent stress, p_e, for the failure line given by:

$$q = Mp_e \tag{12}$$

where $p_e = \sqrt{(p - u_a)^2 + (u_a - u_w)^2}$ and M is a function of p_e and $(u_a - u_w)/(p - u_a)$.

The above equations for the failure line or critical state line highlighted that the relationship in the q-p space for unsaturated soil must account for suction. If presented in the q-p-s space, the failure line is a non-linear surface. The simplicity of Equation 4 is lost in the suggested relations given by Equations 6 to 12 due to the fact that the associated M coefficients are functions of stresses, degree of saturation and soil fabric. The dependency of the M coefficients with degree of saturation in Equations 6, 7 and 10 should be discouraged as a stress versus volume- mass constitutive relation is mixed with a stress versus stress constitutive relation. A more general relationship for the failure line would take the following form:

$$q = Mf(p - u_a, u_a - u_w) \tag{13}$$

where M is a material characteristic independent of stresses and $f(p\text{-}u_a, u_a\text{-}u_w)$ is an independent function of $(p - u_a)$ and $(u_a - u_w)$. Equation 13 can be extended to provide a more consistent form with Equation 12:

$$q = M(p - u_a)\left[1 + \left(\frac{u_a - u_w}{p - u_a}\right)^n\right]^{1/m} = Mp_e \tag{14}$$

where M, n and m are material characteristics. In Equation 12, $n = m = 2$. The applicability of Equation 14 is illustrated in Figures 2 and 3. The implication of $n \neq m$ needs to be further explored as this will cause a problem on the $(p\text{-}u_a) = 0$ plane.

The yield surface is an ellipse in the modified Cam clay model and can be described by:

$$q^2 - M^2 p'(p'_c - p') = 0 \tag{15}$$

where p'_c is the preconsolidation stress.

Alonso et al. (1990) modified Equation 15 for unsaturated soil as:

$$q^2 - M^2[(p - u_a) + k(u_a - u_w)][(p_c - u_a) - (p - u_a)] = 0 \tag{16}$$

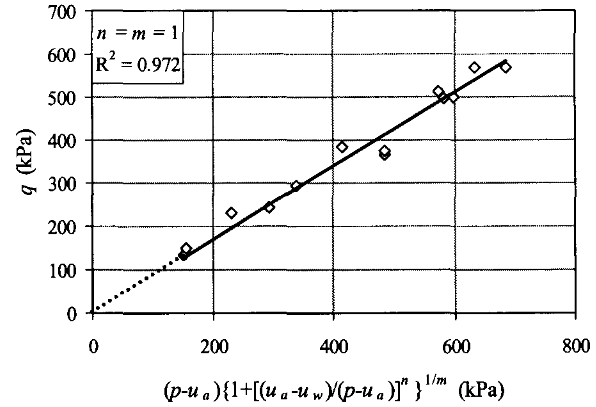


Figure 2. Application of Equation 14 to data of Alonso et al. (1990).

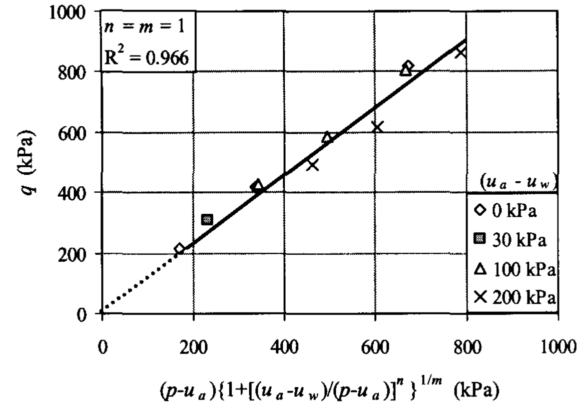


Figure 3. Application of Equation 14 to data of Wang et al. (2002).

Wheeler & Sivakumar (1995) suggested a yield curve which ends at the critical state line being represented by the following equation:

$$\begin{aligned} q^2 - M_*^2[(p - u_a)_0 - (p - u_a)] \\ [(p - u_a) + (p - u_a)_0 - 2p_x] = 0 \end{aligned} \tag{17}$$

where $(p\text{-}u_a)_0$ is the $(p\text{-}u_a)$ value of the yield curve on the $(p\text{-}u_a)$ axis and p_x = $(p\text{-}u_a)$ value at the point of intersection of the yield curve and the critical state line and M_* is the aspect ratio of the ellipse given by:

$$M_* = \frac{M_s p_x + \mu_s}{(p - u_a)_0 - p_x} \tag{18}$$

Sun et al. (2000) used the following yield function:

$$q - M_s p \ln\left(\frac{p_y + \frac{s}{1 + s/a}}{p}\right) = 0 \tag{19}$$

which is a modified form of the original Cam clay yields surface. Jommi (2000) substituted p' in Equation 15 with the average soil skeleton mean stress, $\hat{p}$, to give:

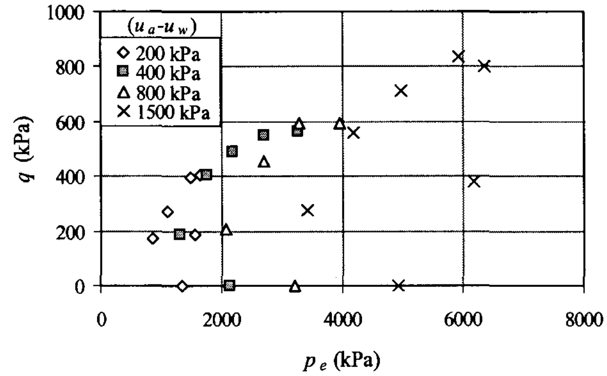
$$q^2 - M^2\hat{p}\left[\hat{p}_0 - (p-u_a)\right] = 0 \qquad (20)$$

where $\hat{p} = \left[(p-u_a) + S_r(u_a-u_w)\right]$. Cui & Delage (1996) suggested using an ellipse inclined along the K_0 line as the yield curve for an unsaturated compacted silt. The ellipse is given by:

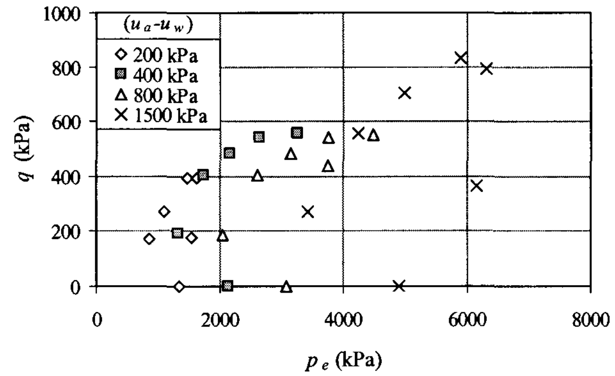
$$b^2\left[(p-u_a)\cos\theta + q\sin\theta - c\right]^2$$

$$+ a^2\left[-(p-u_a)\sin\theta + q\cos\theta\right]^2 - a^2b^2 = 0 \qquad (21)$$

where a and b are the major and minor axes of the ellipse, c defines its position and θ defines its inclination. Tang and Graham (2002) suggested yield curves that are ellipses in the stress ratio, $\eta_s[=(u_a-u_w)/(p-u_w)]$, planes. For each p-s stress state, an equivalent preconsolidation pressure, p_{ec}, can be obtained from the loading collapse curve. The q-p_e plot when normalised using p_{ec} collapses the yield curves representing different suctions into a single yield curve. The idea presented by Tang & Graham is explored using the experimental data of Cui & Delage (1996). The experimental data are replotted on a q-p_e plot where p_e is as given in Equation 14 (Fig. 4). The p_e in Figure 4 was obtained using a value of 0.4 for n and m. The normalized plots are then obtained by normalizing p_e with p_{e0} and q with p_{ec} and shown as Figure 5.

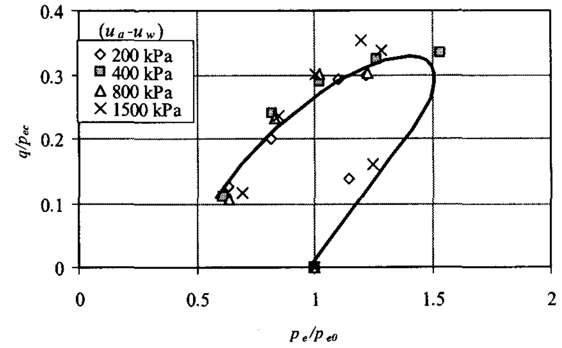


(a) $M_k = 1.85$

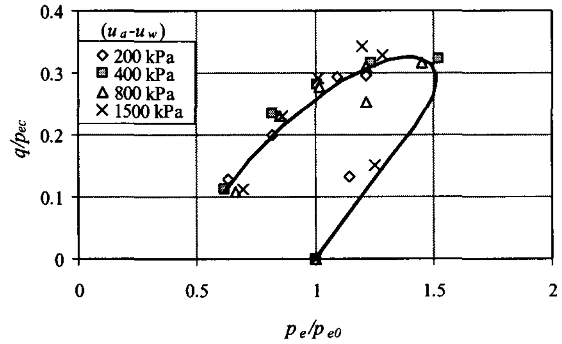


(b) $M_k = 2.1$

Figure 4. Data from Cui & Delage replotted on q-p_e plot.



(a) $M_k = 1.85$



(b) $M_k = 2.1$

Figure 5. Normalised q- p_e plots of Figure 4.

In the p-s space, Alonso et al. (1990) proposed a loading-collapse (LC) yield curve and a suction increase (SI) line as the limiting boundaries (Fig. 6). The LC yield curve is described using the following equation:

$$\left(\frac{p_0}{p^c}\right) = \left(\frac{p_0^*}{p^c}\right)^{(\lambda_s - \kappa/\lambda_0 - \kappa)} \qquad (22)$$

where p_0 are $(p-u_a)$ values on the LC yield curve, p^c is the $(p-u_a)$ value at which it is possible to reach the saturated virgin state starting from an unsaturated condition through a wetting path which involves only elastic swelling, p_0^* is the p_0 value on the p_0 axis, λ_s is the slope of the normal compression line for a suction s, λ_0 is the λ_s value for $s = 0$ and κ is the slope of the swelling line. The λ_s value is given by the following equation:

$$\lambda_s = \lambda_0\left[(1-r)\exp(-\beta s) + r\right] \qquad (23)$$

where r is a constant relating to the maximum stiffness of the soil and β is a parameter which controls the rate of increase of soil stiffness with suction. Another form of Equation 23 was proposed by Kohgo et al. (1993):

$$\lambda_s = \frac{\lambda_0}{1 + \left(\frac{s - s_e}{a}\right)^p} \qquad (24)$$

where a and p are material parameters. Both equations 23 and 24 can be made to give similar curves with appropriate values for the parameters.

According to Tang & Graham (1996), the p_{ec} points are obtained from the loading collapse (LC) curves given the p-s stress state. These data are not available from Cui & Delage. In Figure 5, the p_{ec} values were fitted values such that the yield curves collapsed into a single curve. The fitted p_{ec} values and the corresponding suction values of the yield curve are plotted in Figure 7 and showed a similar shape as the loading collapse curve. Surprisingly, both sets of fitted p_{ec} values fall within a single curve.

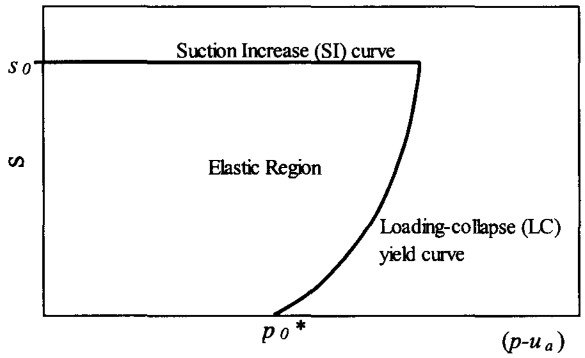


Figure 6. Loading-collapse and suction increase yield curves.

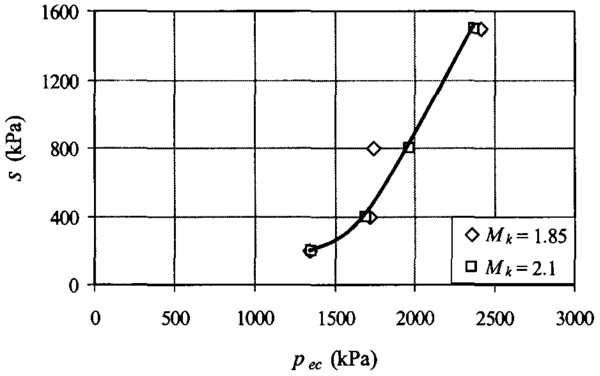


Figure 7. Relationship between s and p_{ec}.

3.2 *The e-p-s space*

The compression behaviour of saturated soils has been historically presented on an e-log p' plot. In the critical state soil model, the normal compression line is given by:

$$e = e_N - \lambda_0 \ln p' \qquad (25)$$

and the critical state line is given by:

$$e = e_{cs} - \lambda_0 \ln p' \qquad (26)$$

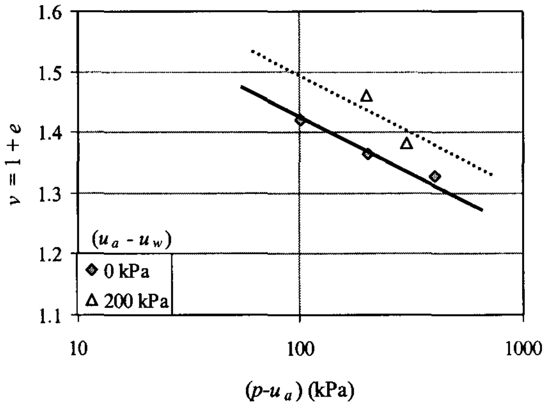
Sometimes, e in Equations 25 and 26 is expressed as specific volume, v, where $v = 1+e$.

For unsaturated soils, it was found that the slope of the compression line, λ_s, is dependent on suction. Alonso et al. (1990) proposes that λ_s decreases monotonically with suction whereas Wheeler & Sivakumar (1995) showed experimental data of compacted kaolin where λ_s generally increases with suction. Generally, one would expect λ_s to decrease with suction as the soil becomes increasing stiffer as its suction increases. The increase in shear stiffness with suction has been observed by Mancuso et al. (2000). The expressions for λ_s as proposed by Alonso et al. (1990) and Kohgo et al. (1993) are given by Equations 23 and 24, respectively.

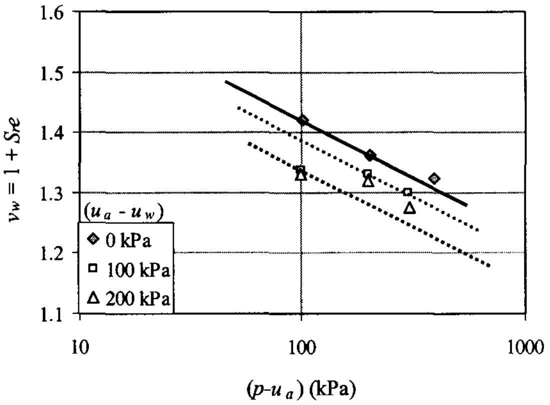
Wang et al. (2002) suggested that the specific water volume, v_w, be used as one of the critical state variables for unsaturated soil. The specific water volume is defined as:

$$v_w = 1 + S_r e \qquad (27)$$

When specific water volume is used instead of specific volume, the critical state lines corresponding to higher suctions are below the critical state line for the saturated condition as illustrated in Figure 8.



(a) Critical state lines on $v - (p\text{-}u_a)$ plane



(b) Critical state lines on $v_w - (p\text{-}u_a)$ plane

Figure 8. Critical state lines on $v - (p\text{-}u_a)$ and $v_w - (p\text{-}u_a)$ planes for "Botkin silt" data from Wang et al. (2002).

Figure 8 would suggest that it is now possible to plot specific water volume, v_w, against the equivalent stress, p_e, to obtain a single line as illustrated in Figure 9. It is also noted that Pandian et al. (1992) and

Herkal et al. (1995) have pointed out that the normal compression lines for different suction values can be reduced to a single line by plotting $e\sqrt{S_r}$ versus log p.

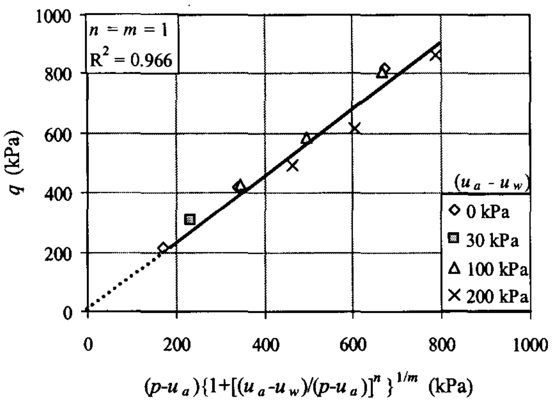


Figure 9. Critical state line on v_w-p_e plot.

4 CONCLUSION

The development of constitutive models is important for the understanding of soil behaviour as well as for the solution of engineering problems. Stress versus stress and stress versus volume-mass constitutive relations developed for unsaturated soil models based on elasto-plasticity have been examined. It is suggested that stress-state variables should not be mixed with volume-mass variables in deriving an equivalent "effective stress". Not all the features of an elasto-plastic constitutive model for unsaturated soils are examined in the present paper due to length constraint. However, the lack of complete experimental data sets to develop and to verify constitutive models for unsaturated soils remains the greatest task at hand.

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