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USING VOLUME CHANGE INDICES FOR TWO-DIMENSIONAL SWELLING ANALYSIS

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ABSTRACT
This paper presents a finite element model that can be used to simulate swelling in two- and three-dimensions. The model formulation is based on the general theory of unsaturated soil behaviour. The elasticity parameter functions are computed from volume change indices. It is necessary to assume a value for Poisson's ratio. Soil suction conditions in a soil profile are predicted with time using a saturated/unsaturated seepage analysis. Solutions of the volume change problems are obtained using a general partial differential equation solver, called FlexPDE[1]. A typical two-dimensional problem associated with deformations in an expansive soil resulting from a transient wetting process into the soil is presented to demonstrate the applicability of the proposed model.

RÉSUMÉ
C'est article présente un modèle d'élément fini qui peut être utilisé pour simuler le gonflement dans deux- et trois-dimensions. La formulation du modèle est basée sur la théorie générale du comportement de sol non saturé. Les fonctions de paramètre d'élasticité sont calculées des index de changement de volume. C'est nécessaire d'assumer une valeur pour le coefficient de Poisson. Les conditions de succion avec le temps dans un profil de sol sont prédites avec l'utilisation d'une analyse d'écoulement de l'eau saturé/non saturé. Solutions de changement de volume sont obtenues en utilisant un solutionneur général d'équation partiel différentiel, appelé FlexPDE. Un exemple typique à deux dimensions associé à déformations dans un sol expansif qui résulte d'un processus de mouillage dans le sol est présenté pour démontrer la validité d'application du modèle proposé.

1. INTRODUCTION

Lightly loaded structures constructed in unsaturated, expansive soils are often subject to severe heave and/or distress as a result of pore-water pressure changes in the soil. Changes in pore-water pressure may be caused by changes in the depth of the water table, a reduction in natural evaporation from the ground surface, water uptake by vegetation, excessive watering of lawns, and leakage of underground water and sewer lines.

Numerous methods for the prediction of heave have been proposed in the literature. These methods generally make use of a linear relationship between void ratio and the logarithm of matric suction (or net normal stress). Heave problems in engineering practice are often two-dimensional or three-dimensional in character (i.e., roadways, airport runways or houses) and the suggested methods of analysis can only be used to predict one-dimensional heave. There are several computational problems that need to be solved in order to move from a one-dimensional analysis to a two- or three-dimensional analysis. These problems include the evaluation and substitution of elasticity parameter functions into a finite element numerical model. The solution becomes a highly non-linear numerical problem.

This paper presents a finite element model that can be used to simulate swelling in two- and three-dimensions. The model formulation is based on the general theory of unsaturated soil behaviour. The elasticity parameter functions required in the constitutive relationship are computed from volume change indices obtained from one-dimensional, oedometer tests that are subsequently converted to an elasticity parameter that is a function of the stress state. It is necessary to assume a value for Poisson's ratio. Soil suction conditions in a soil profile are predicted with time using a saturated/unsaturated seepage analysis.

Solutions of the volume change problems are obtained using a general partial differential equation solver, called FlexPDE. A typical two-dimensional problem associated with deformations in an expansive soil resulting from water infiltration into the soil is presented to demonstrate the applicability of the proposed model.

2. VOLUME CHANGE THEORY OF UNSATURATED SOILS

2.1 Constitutive Relations

Two stress state variables are needed to describe volume change behavior of an unsaturated soil (Fredlund and Morgenstern, 1977). These stress state variables are net normal stress, $(\sigma-u_a)$, and matric suction, (u_a-u_w), where σ is total normal stress, u_a is pore-air pressure and u_w is pore-water pressure. With the use of these two stress state variables, volume changes in the soil due to externally applied loads and the environmental changes (i.e., change in groundwater table, water uptake by a tree or infiltration) can be considered separately.

Assuming the soil behaves in an incrementally isotropic, linear elastic material, the soil structure constitutive relations can be written as follows (Fredlund and Rahardjo, 1993):

$$d\varepsilon_{ij} = \frac{1+\mu}{E} d(\sigma_{ij} - \delta_{ij} u_a) - \frac{\mu}{E} d(\sigma_{kk} - 3u_a)\delta_{ij} + \frac{d(u_a - u_w)}{H}\delta_{ij} \quad [1]$$

where: ε_{ij} = components of the strain tensor for the soil structure, σ_{ij} = components of the total stress tensor for the soil structure, σ_{kk} = $(\sigma_{11} + \sigma_{22} + \sigma_{33})$, δ_{ij} = the Kronecker delta, μ = Poisson's ratio, E = elasticity parameter for the soil structure with respect to a change in net normal stress, and H = elasticity parameter for the soil structure with respect to a change in matric suction.

The constitutive relationships showed in Eq. 1 can be used to solve non-linear elastic volume change problems associated with unsaturated soils in two- or three-dimensions.

2.2 Calculation of elasticity parameters from volume change indices

The constitutive surface for the soil structure of a swelling soil can be obtained when void ratio is plotted with respect to the logarithms of the stress state variables (Fig. 1). The logarithmic plots are essentially linear over a relatively large stress range on the extreme planes (Fredlund and Rahardjo, 1993). Slopes of the void ratio versus logarithm of net normal stress or matric suction lines are called volume change indices, C_t or C_m (Fig. 1). These indices are used in many methods of one-dimensional heave prediction.

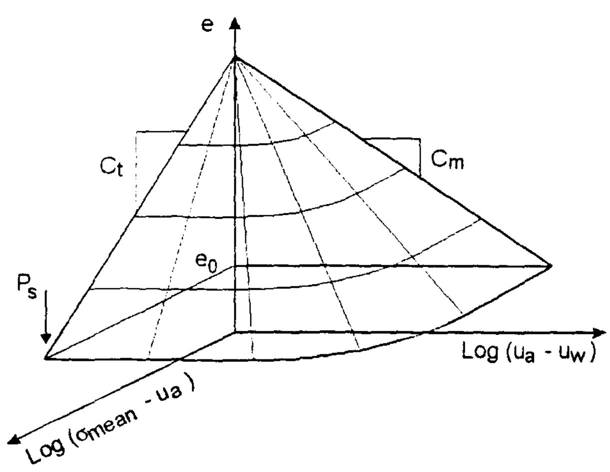

Figure 1. Semi-logarithmic plot of void ratio constitutive surface

Using a mathematical conversion between a semi-logarithmic scale and arithmetic scale, the coefficients of volume change can be written in term of the volume change indices as follows (Hung and Fredlund, 2000):

$$m_1^s = \frac{0.434 C_t}{1+e_0} \frac{1}{(\sigma_y - u_a)} \quad [2]$$

$$m_2^s = \frac{0.434 C_m}{1+e_0} \frac{1}{(u_a - u_w)} \quad [3]$$

where: m_1^s = coefficient of volume change of soil structure with respect to changes in net normal stress, m_2^s = coefficient of volume change of soil structure with respect to changes in matric suction, e_0 = initial void ratio, C_t = volume change index with respect to net normal stress, C_m = volume change index with respect to matric suction, and $(\sigma_y - u_a)$ = net total vertical stress.

The elasticity parameters, E and H, can be related to the coefficients of volume change for K_0-loading (i.e., oedometer test) and Poisson's ratio as follows (Fredlund and Rahardjo, 1993):

$$E = \frac{(1+\mu)(1-2\mu)}{m_1^s(1-\mu)} \quad [4]$$

$$H = \frac{(1+\mu)}{m_2^s(1-\mu)} \quad [5]$$

These elasticity parameters can be calculated for K_0-loading from the volume change indices, initial void ratio and Poisson's ratio by substituting Eqs. 2 and 3 into Eqs. 4 and 5, respectively.

$$E = \frac{(1+\mu)(1-2\mu)}{(1-\mu)} \frac{(1+e_0)}{0.434 C_t} (\sigma_y - u_a) \quad [6]$$

$$H = \frac{(1+\mu)}{(1-\mu)} \frac{(1+e_0)}{0.434 C_m} (u_a - u_w) \quad [7]$$

Equations 6 and 7 can be written for plane strain (i.e., two-dimensional) loading conditions as follows (Hung and Fredlund, 2000):

$$E = \frac{4.605(1+\mu)(1-2\mu)(1+e_0)}{C_t} (\sigma_{ave} - u_a) \quad [8]$$

$$H = \frac{4.605(1+\mu)(1+e_0)}{C_m} (u_a - u_w) \quad [9]$$

where: $\sigma_{ave} = (\sigma_x + \sigma_y)/2$, average total normal stress for two-dimensional loading.

3. GOVERNING PARTIAL DIFFERENTIAL EQUATION FOR SATURATED/UNSATURATED WATER FLOW

In stress-deformation analyses, displacements are calculated from changes in stress states (i.e., changes in net normal stress and matric suction) and the elasticity parameters. Matric suction profile is necessary to describe the initial and final stress conditions in soils. If the pore-air pressure is assumed atmospheric, the distribution of pore-water pressure is equivalent to the matric suction distribution. The prediction of pore-water pressure must take into consideration in response to changes in the surface flux boundary conditions (i.e., infiltration, evaporation, and evapotranspiration) and the fluctuation of the ground-water tables. The pore-water pressure distribution within the soil can be estimated by performing a saturated-unsaturated seepage analysis.

The governing partial differential equation for water flow through a heterogeneous, anisotropic, saturated-unsaturated soil can be derived by satisfying conservation of mass for a representative elemental volume, assuming that flow follows Darcy's law with a non-linear coefficient of permeability. If it is assumed that the total stress remains constant during a transient process and that pore-air pressure is atmospheric, the differential equation can be written as follows for the two-dimensional transient case (Thieu *et al.*, 2000):

$$\frac{\partial}{\partial x}\left(k_x \frac{\partial h}{\partial x}\right)+\frac{\partial}{\partial y}\left(k_y \frac{\partial h}{\partial y}\right)=m_2^w \gamma_w \frac{\partial h}{\partial t} \qquad [10]$$

where: h = total head (i.e., pore-water pressure head plus elevation head); k_x and k_y = coefficient of permeability of the soil in the x- and y-direction, respectively; γ_w = the unit weight of water (i.e., 9.81 kN/m^3), m_2^w = the slope of the soil water characteristic curve.

Both the coefficient of permeability and coefficient of water storage are dependent on stress states in soils (i.e., net normal stress and matric suction). However, these coefficients of an unsaturated soil are predominantly a function of the matric suction.

The water storage indicates the amount of water taken or released by the soil because of a change in the pore-water pressure and can be represented by the slope of the soil-water characteristic curve. Therefore, the water storage function is obtained by differentiating the soil-water characteristic curve with respect to matric suction. Numerous equations have been proposed to simulate the soil-water characteristic curve (Gardner, 1958; van Genuchten, 1980; Fredlund and Xing, 1994). The soil-water characteristic curve described in the present study is limited to the Fredlund and Xing (1994) equation. The Fredlund and Xing (1994) equation is shown below:

$$\theta = \theta_s \left[\frac{1}{\ln\left(e + (\psi/a)^n\right)}\right]^m \qquad [11]$$

where: ψ = soil suction (kPa), e = natural log base, 2.71828..., θ_s = volumetric water content at saturation, a = a soil parameter which is related to the air entry value of the soil (kPa), n = a soil parameter which controls the slope at the inflection point in the soil-water characteristic curve, and m = a soil parameter which is related to the residual water content of the soil.

There are several permeability functions that have been proposed to represent the permeability function of an unsaturated soil (e.g., Gardner, 1958; Fredlund and Xing, 1994; Leong and Rahardjo, 1997). These equations involve finding best-fit parameters, which produces a curve that fits the measured data. The equation proposed by Leong and Rahardjo (1997) is used to describe the permeability function for transient water flow analysis in this paper. Leong and Rahardjo (1997) illustrated that the coefficient of permeability is a power function of volumetric water content. Using the Fredlund and Xing (1994) equation, the permeability function was shown to take the following form:

$$k = k_s \left[\frac{1}{\ln\left(e + (\psi/a)^n\right)}\right]^{mp} \qquad [12]$$

The parameter p can be determined by using a curve fitting of the coefficient of permeability data. The slope of soil-water characteristic curve (i.e., coefficient of water storage) is obtained by differentiating the Fredlund and Xing (1994) equation (Fredlund, 1995).

The transient water flow equation (Eq. 10) along with the equation of a soil-water characteristic curve (Eq. 11) and a permeability function (Eq. 12), can be used to predict pore-water pressure profiles (i.e., suction profiles) at different times during a seepage process. The suction profiles can then be used to compute the suction change for the stress-deformation analysis. The deformations due to changes in suction during any time period can then be predicted by specifying the initial and final soil suction profile.

4. GOVERNING PARTIAL DIFFERENTIAL EQUATION FOR STRESS-DEFORMATION PROCESSES

The governing partial differential equations in term of displacements in x- and y-direction (i.e., u and v) for plane strain loading ($d\varepsilon_z$ = 0) of an isotropic, non-linear elastic soil can be written as follows (Hung and Fredlund, 2000):

$$\frac{\partial}{\partial x}\left\{c\left[(1-\mu)\frac{\partial u}{\partial x}+\mu\frac{\partial v}{\partial y}-\frac{(1+\mu)}{H}(u_a-u_w)\right]\right\}+\frac{\partial}{\partial y}\left\{G\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)\right\}=0 \qquad [13]$$

$$\frac{\partial}{\partial x}\left\{G\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)\right\}+\frac{\partial}{\partial y}\left\{c\left[\mu\frac{\partial u}{\partial x}+(1-\mu)\frac{\partial v}{\partial y}-\frac{(1+\mu)}{H}(u_a-u_w)\right]\right\}+\rho g=0 \qquad [14]$$

where: $c=\frac{E}{(1-2\mu)(1+\mu)}$, $G=\frac{E}{2(1+\mu)}$, ρ = density of the soil, and g = acceleration due to gravity.

Equations 13 and 14 can be used to compute the displacements in horizontal and vertical directions under an applied load or due to changes in matric suction. These equations can also be used to compute the induced stresses in the soil under an applied load. Because of soil property non-linearity, an incremental procedure is used to obtain the solution of these equations. In the incremental procedure, the values of elasticity parameters E and H are assumed to be unchanged within each stress and strain increment, but are changed from one loading increment to another.

5. FLEXPDE COMPUTER PROGRAM

A finite element computer program, called FlexPDE, marketed by PDE Solutions Inc. is one of several available general partial differential equation solvers. FlexPDE can be used to solve both water flow and stress-deformation problems in two- or three-dimensions. The user must specify the governing partial differential equations to be solved (such as Eq. 10 and Eqs. 13, 14). The material properties can be described in the tabular form or as a mathematical equation (Fredlund et. al., 2000). Boundary conditions can be specified as a dependent variable type or a derivative of a dependent variable type.

This software package has several special features that are of interest to geotechnical engineers. Major features of these newly developed solvers include:

- AutoCAD style CAD input
- automatic mesh generation and refinement
- adaptive time step design and refinement
- ensuring convergence when solving non-linear equations
- allowing material properties to be input in a variety of forms
- input 3D problems as surfaces and layers using survey data.

6. EXAMPLE PROBLEM

This example problem considers a 5 metre deep deposit of an unsaturated swelling clay that rests underneath a flexible cover (Fig. 2). Swelling due to leakage of water under the cover will be predicted for various predetermined elapsed times. The initial matric suction in the soil mass was assumed to be constant and with a value equal to 400 kPa.

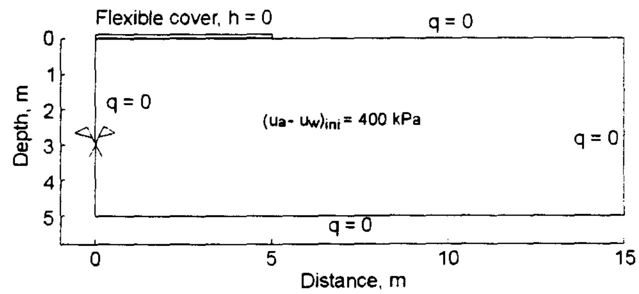

Figure 2. Illustration of example problem and boundary conditions for seepage analysis

The following soil properties are assumed for analysis:

Table 1. Assumed soil properties for swelling analysis

Soil proprties	Values
Initial void ratio, e_0	1.0
Swelling index, C_m	0.1
Poisson's ratio, μ	0.35
Coefficient of permeability at saturation, k_s	1 mm/day
Volumetric water content at saturation, θ_s	0.45
Initial matric suction	400 kPa
Parameters for SWCC [Fredlund and Xing (1994)] and permeability function [Leong and Rahardjo (1997)]	a = 100 kPa n = 1.5 m = 1 p = 1

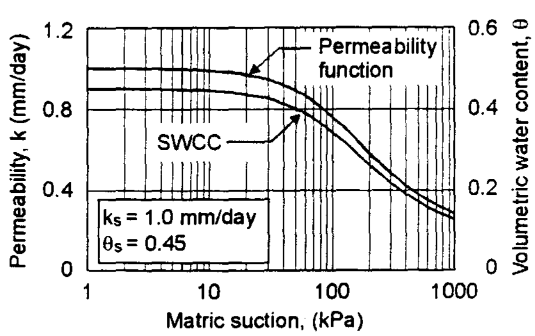

Figure 3. Permeability function and SWCC

The soil-water characteristic curve is described using the Fredlund and Xing (1994) equation, with the parameters *a* equal to 100 kPa, *n* equals 1.5 and *m* equals 1. The permeability function is described using the equation proposed by Leong and Rahardjo (1997) based on the Fredlund and Xing (1994) equation for the soil-water characteristic curve with *p* equal to 1. The soil-water characteristic curve and the permeability function have the same shape in this example (Fig. 3).

The elasticity parameter with respect to matric suction, H, can be calculated from given swelling index with respect to matric suction, C_m, initial void ratio, e_0, and assumed Poisson's ratio, μ using Eq. 9. This function can be written as follows:

$$H = 124.335(u_a - u_w) \quad [15]$$

Transient water flow analysis is performed to predict matric suction conditions in the soil at various elapsed times (i.e., at days 10, 25, 50, 100, 150, and 200). Stress-deformation analysis is then performed to predict displacements due to changes in matric suction.

6.1 Water flow analysis

Boundary conditions for water flow analysis is presented in Fig. 2. The transient wetting process was induced by imposing a pore-water pressure equal to 0 kPa at the ground surface of the soil deposit under the flexible cover. Such boundary condition simulates the water infiltration into the soil mass due to leakage through the cover. Flux equal to zero is specified elsewhere on the boundary. The analysis is conducted to predict matric suction changes with time as the wetting front advances into the soil mass.

The matric suction distributions at days 25 and 100 are presented in Figs. 4 and 5, respectively. Most of suction changes in the first 25 days occurred near ground surface below the cover. Development of matric suction at some points in the soil profile is presented in Fig. 6. It can be noted that suction changed rapidly under the cover in the first 50 days. Development of matric suction below the cover versus time is shown in Fig. 7. After 200 days of wetting, matric suction below the cover reduced to about 50 kPa.

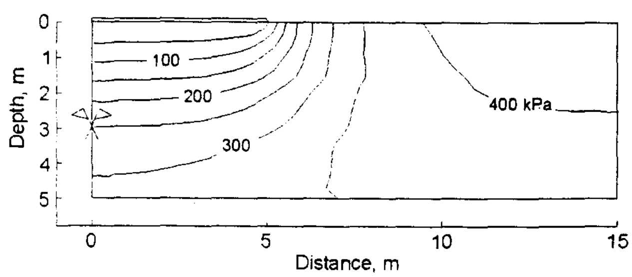

Figure 4. Matric suction distribution at day 25

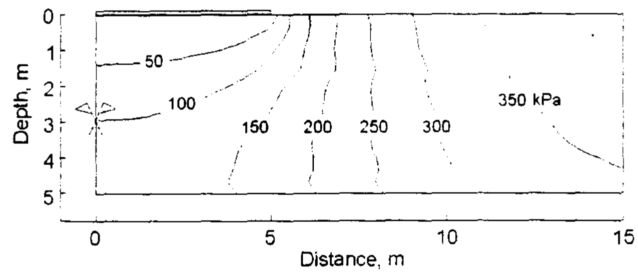

Figure 5. Matric suction distribution at day 100

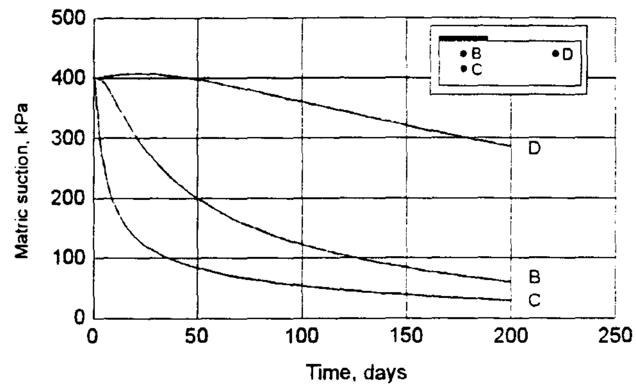

Figure 6. Development of matric suction for points B, C, and D

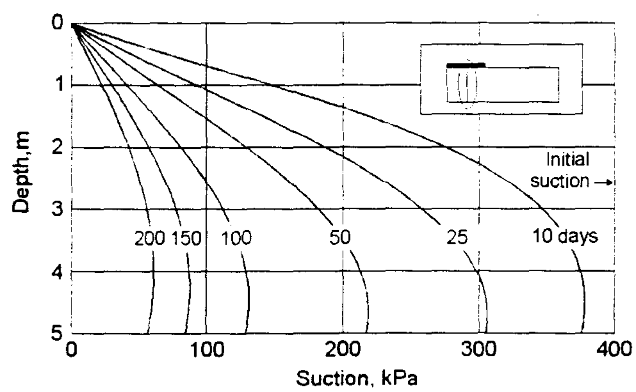

Figure 7. Development of matric suction below the cover versus depth with time

6.2 Stress-deformation analysis

Boundary conditions for stress-deformation analysis is presented in Fig. 8. The surface of the domain has no restriction in term of displacements. The domain is prevented from movement in the horizontal direction at both left (by symmetry) and along the right limits. The bottom limit is prevented from moving in both the horizontal and vertical directions.

Contours of horizontal and vertical displacements at day 25 after wetting commences are presented in Figs. 9 and 10, respectively. Contours of horizontal and vertical displacements at day 100 are presented in Figs. 11 and

12, respectively. Most of the heave occurred below the cover in the first 100 days, where the changes in matric suctions are the largest.

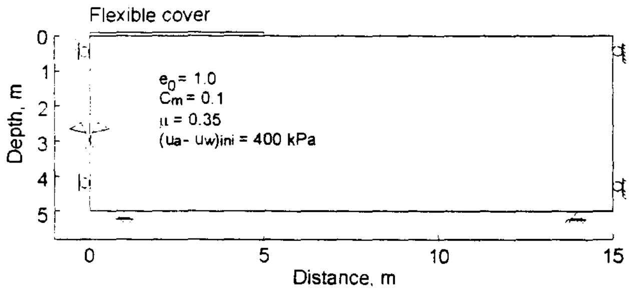

Figure 8. Boundary conditions for stress analysis

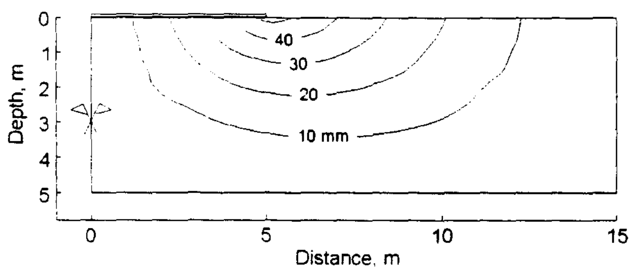

Figure 9. Contours of horizontal displacements at day 25

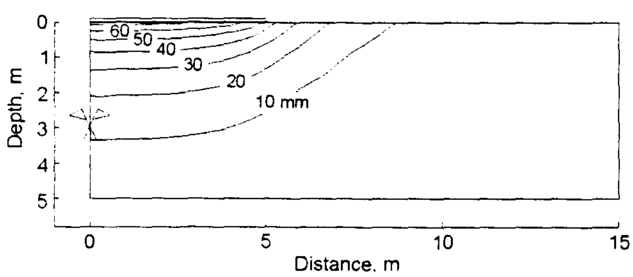

Figure 10. Contours of vertical displacements at day 25

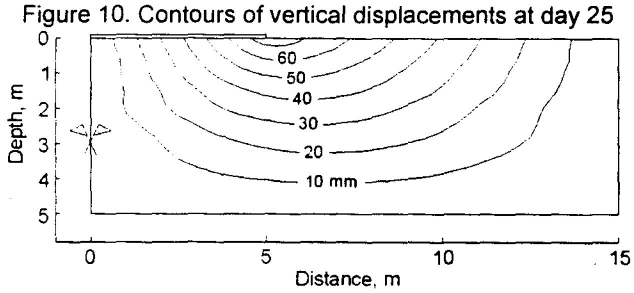

Figure 11. Contours of horizontal displacements at day 100

Figure 13 shows the development of heave at some points under the cover. Figures 14 and 15 present the cumulative heave versus depth and along the ground surface, respectively. Maximum heave of 200 mm is predicted at the cover location for day 200. Significant differential heave of 70 mm is predicted from center to the edge of the cover at day 200.

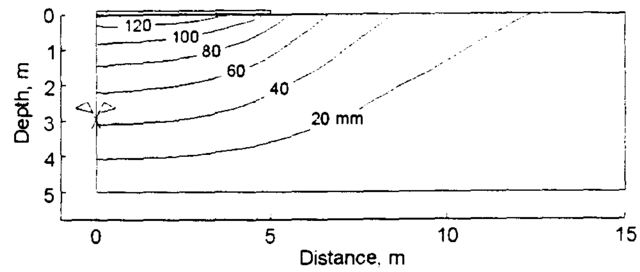

Figure 12. Contours of vertical displacements at day 100

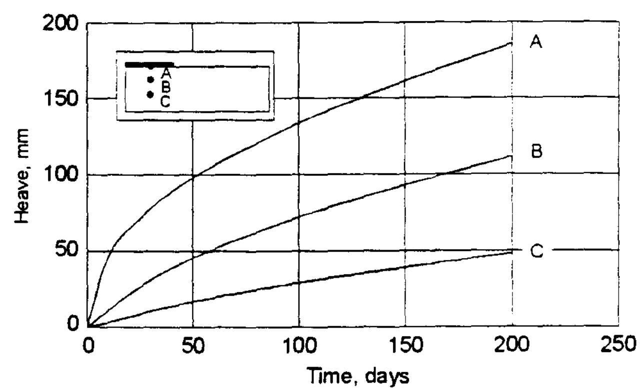

Figure 13. Development of heave for points A, B, and C

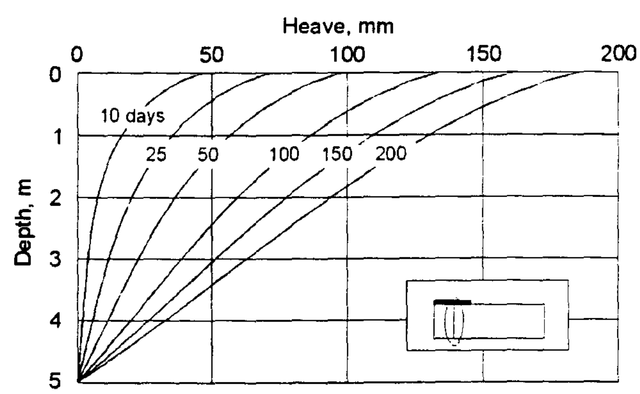

Figure 14. Development of heave below the cover versus depth with time

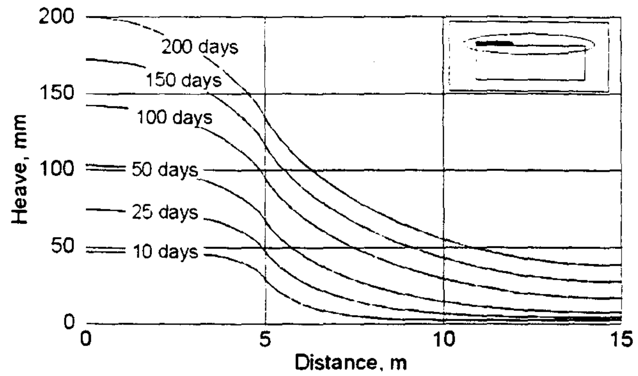

Figure 15. Development of heave versus time at the ground surface

7. CONCLUSION

The volume change indices, which are obtained from one-dimensional test results, can be converted to the elasticity parameters for two- and three-dimensional swelling analysis. A value of Poisson's ratio must be assumed.

Partial deferential equation solver, FlexPDE, can be used with non-linear elastic parameter functions to predict volume change in unsaturated, swelling soil system in two-dimensions.

REFERENCE

Fredlund D.G., and Morgenstern, N.R. 1977. Stress State Variables for Unsaturated Soils. ASCE Journal of Geotechnical Engineering, **103**: 447-466.

Fredlund, D.G. and Xing, A. 1994. Equation for the Soil-Water Characteristic Curve. Canadian Geotechnical Journal, **31**(3): 521-532.

Fredlund, D.G. and Rahardjo, H. 1993. Soil Mechanics for Unsaturated Soils. John Wiley & Sons, New York, 560 p.

Fredlund, D.G., Hung, V.Q., and Thieu, N.T.M. 2000. A Component System Approach to Solving Saturated-Unsaturated Soils Problems. Proceeding, Annual Conference of the Canadian Society for Civil Engineering, London, Ontario, Canada, pp. 448-455.

Gardner, W.R. 1958. Some Steady State Solutions of Unsaturated Moisture Flow Equation with Application to Evaporation from a Water Table. Soil Science, **85**: 228-232.

Hung, V.Q. 2000. Finite Element Method for the Prediction of Volume Change in Expansive Soils. M.Sc. Thesis, University of Saskatchewan, Saskatoon, SK, Canada, 314 p.

Hung, V.Q. and Fredlund, D.G. 2000. Volume Change Predictions in Expansive Soils Using a Two-dimensional Finite Element Method. Proceeding, Asian Conference on Unsaturated Soils, Singapore, pp. 231-236.

Leong, E.C. and Rahardjo, H. 1997. Permeability Functions for Unsaturated Soils. Journal of Geotechnical and Geoenvironmental Engineering, ASCE, pp: 1118-1126.

Thieu, N.T.M, Fredlund, D.G., and Hung, V.Q. 2000. General Partial Differential Equation Solvers for Saturated-Unsaturated Seepage. Proceeding, Asian Conference on Unsaturated Soils, Singapore, pp. 201-206.

van Genuchten, M.T. 1980. A Closed-formed Equation for Predicting the Hydraulic Conductivity of Unsaturated Soils. Soil Science Society of America Journal, **44**: 892-898.

[1]FlexPDE is a propriety product of PDE Solutions Inc., 2120 Spruce Way, Antioch, CA 94509, USA