SLOPE STABILITY ANALYSIS USING DYNAMIC PROGRAMMING COMBINED WITH FINITE ELEMENT STRESS ANALYSIS

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ABSTRACT: A new method for slope stability analysis is presented. The method combines the theory of dynamic programming with a finite element stress analysis to search for the critical slip surface in a slope. The critical slip surface is defined as the path through which the minimum value of the 'optimal function' is obtained. The only assumption regarding the shape of the critical slip surface is that the surface is an assemblage of linear segments. A computer program called DYNPROG is developed based on the dynamic programming theory. Two examples have been analyzed through the use of DYNPROG in order to verify the proposed method. Results show that the new method provides satisfactory solution in terms of the location of the slip surface and the value of the factor of safety.

***Key words**: slope stability, dynamic programming, optimization techniques, finite element method*

1. INTRODUCTION

A slope stability analysis consists of the calculation of the factor of safety for a specified slip surface and the determination of the critical slip surface, which has the lowest factor of safety. In order to render the inherently indeterminate analysis determinate, the early methods (Fredlund and Krahn, 1977) make use of assumptions regarding the location of the slip surface and the distribution of the normal force acting at the base of slices. Accordingly, these methods are only useful for resolving the location of an assumed shape for the slip surface. Since the shape of the critical slip surface is assumed by the analyst, the early methods are somewhat incomplete.

Over the past two decades, there has been wide variety of proposed methods in the literature for completely resolving the problem of slope stability. The proposed methods often involve more sophisticated mathematical equations to automatically generate and search for the critical slip surface. The corresponding factor of safety is calculated as long as the critical slip surface is found. The first methods that do not assume the shape are those that adopt the variational calculus theory (Revilla and Castillo, 1977; Baker and Garber, 1978). Unfortunately, the methods were proved to be incomplete by De Josselin De Jong (1980) due to the fact that one of the functionals used in the variational calculus solution to define the critical slip surface is of "degenerate nature and possesses no minimum" (De Josselin De Jong, 1981).

In 1980, Boutrup and Lovell developed a random searching technique for slope stability analyses. The proposed method automatically generates trial slip surfaces that can be circular or irregular in shape. The random searching technique (Boutrup and Lovell, 1980; Greco, 1996) has been considered to be effective only in limited cases. Recently, the attention of some researchers has been directed to the applicability of optimal techniques in slope stability analysis. Celestino and Duncan (1981) used an alternating variable method in combination with Spencer's method; Nguyen (1985) used the simplex method; Chen and Shao (1987) used simplex, steepest descent and Davidon-Fletcher-Powell (DFP) methods to search for the minimum factor of safety and the associated "critical slip surfaces". However, it is noted that these methods merely yield a local rather than a global minimum factor of safety (Greco, 1988).

Research into the application of dynamic programming in slope stability analysis was first published by Baker (1980). In this approach, Baker (1980) coupled the theory of dynamic programming with Spencer's method (Spencer, 1967) to determine the critical slip surface. The method yields the global minimum factor of safety. Yamagami and Ueta (1987) improved on Baker's approach by coupling the dynamic programming search with the finite element method. Zou *et al.* (1995) proposed an improved dynamic programming technique coupled with finite element method to analyze the stability of a trial dam.

The present study is based on the earlier work of Yamagami and Ueta (1987). A computer program called DYNPROG is developed to perform the optimal search using a dynamic programming technique on the stress fields obtained from a finite element stress analysis. The distribution of stresses is determined using a partial general differential equation solver known as FlexPDE.

2. DYNAMIC PROGRAMMING IN SLOPE STABILITY ANALYSIS

(1) Introduction

Dynamic programming is an optimization method, which was first introduced by Bellman (1957). The method can be considered as an optimal scheme for solving multistage optimization problems (Yamagami and Ueta, 1987). Mathematically, dynamic programming itself is not a powerful method since it is applicable only to 'additive functions' (Baker, 1980). However, the method, as presented later, has proved to be suitable particularly for the case of slope stability analyses. In 1980, Baker published a research paper addressing the applicability of dynamic programming in slope stability analysis. Dynamic programming was coupled with Spencer's method (Spencer, 1967) to determine the critical slip surface.

Finite element stress-strain analysis has been extensively used in geotechnical engineering for the estimation of stresses. The main advantage of the finite element method in stress-strain analysis is that the constitutive laws of soils can be readily modelled. Applying the stress distribution obtained from a finite element analysis overcomes some of the shortcoming of limit equilibrium methods regarding the uncertainty of the normal force at the base of slices (Kulhawy, 1969).

In the present method, the dynamic programming technique is performed on the field of finite element stresses. The Kulhawy's (1969) equation for the factor of safety is adopted. Since the constitutive laws of soils are preserved, there is no need to further consider force and moment equilibrium conditions.

(2) General theory

An equation for the overall factor of safety of an arbitrary slip surface AB (Fig. 1) can be defined as:

$$F_s = \frac{\int_A^B \tau_f dL}{\int_A^B \tau dL} \qquad [1]$$

where τ is the mobilized shear stress along the slip surface, τ_f is shear strength of the soil, L is the total length of the slip surface. For the sake of simplicity, the critical slip path is assumed to be an assemblage of linear segments. Each segment connects two 'state' points located in two successive 'stages' (Fig. 2). The stage-state point system forms a grid consisting of rectangle elements. The grid is called the 'searching grid' and the rectangle elements created are called 'grid elements'. In this discretized form, the overall factor of safety can be approximated as:

$$F_s = \frac{\sum_{i=1}^{n} \tau_{f_i} \Delta L_i}{\sum_{i=1}^{n} \tau_i \Delta L_i} \qquad [2]$$

where n is the number of discrete segments, τ_i, τ_{f_i} and ΔL_i are the shear mobilized, shear strength and the length of the i^{th} segment, respectively.

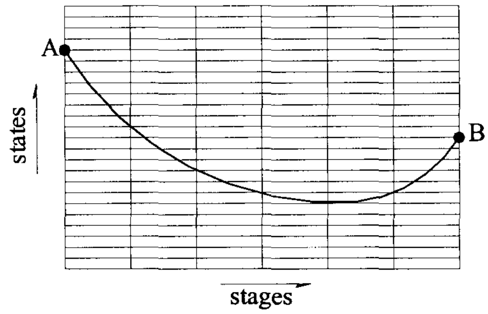

Figure 1. An arbitrary slip path, AB

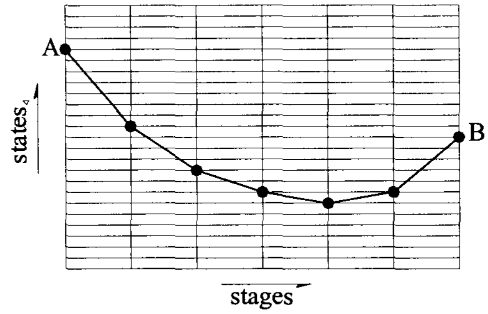

Figure 2. Slip path AB in a discretized form

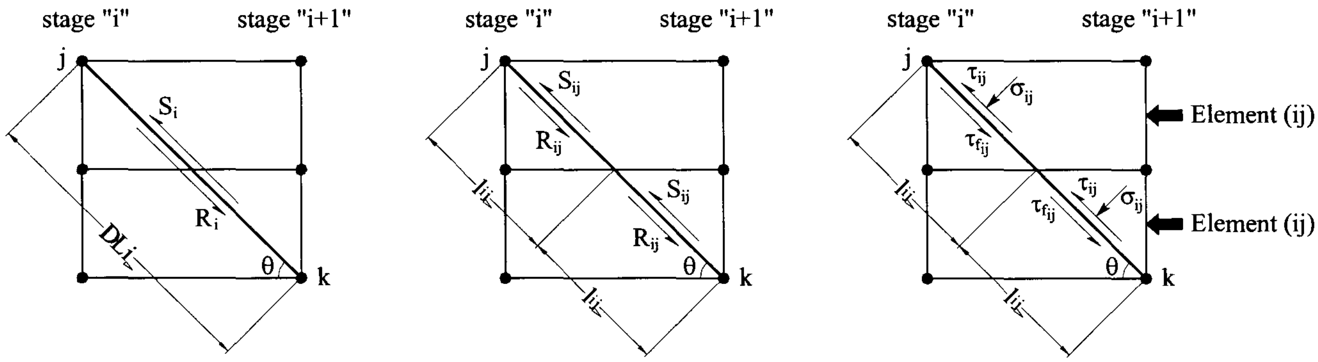

Figure 3. Shear strength and stresses in a discretized form

Since the purpose of the method is to determine the minimum value of the factor of safety, a minimization scheme should be made in order to minimize the value of F_s in [2]. It was shown by Baker (1980) that the minimum of F_s in [2] can be found by minimizing an 'auxiliary function' G (or the so-called 'return function'), which is defined as:

$$G = \sum_{i=1}^{n} (\tau_{fi} - F_s \tau_i)\Delta L_i \qquad [3]$$

Since the auxiliary function has the form of an 'additive function', the dynamic programming technique can be applied to minimize G. Equation [3] can be rearranged as follows:

$$G = \sum_{i=1}^{n} (R_i - F_s . S_i) \qquad [4]$$

where R_i and S_i are the resistance and shear force acting on the i^{th} segment (Fig. 3). The minimum value of the auxiliary function is $G_{\min}$, and:

$$G_{\min} = \min \sum_{i=1}^{n} (R_i - F_s . S_i) \qquad [5]$$

For particular i^{th} segment, the shear strength of soils can be defined using the general theory of unsaturated soil mechanics (Fredlund and Rahardjo 1993):

$$\tau_{f_i} = c' + (\sigma_\theta - u_a)\tan\phi' + (u_a - u_w)\tan\phi_b \qquad [6]$$

where c', ϕ' and ϕ_b are the shear strength parameters of an unsaturated soil, $(\sigma_\theta - u_a)$ is net normal stress and $(u_a - u_w)$ is the soil suction. On the other hand, the mobilized shear stress at a point can be computed from vertical, horizontal and shear stresses:

$$\sigma_\theta = \sigma_x \sin^2\theta + \sigma_y \cos^2\theta - \tau_{xy}\sin 2\theta \qquad [7]$$

$$\tau_\theta = \tau_{xy}(\sin^2\theta - \cos^2\theta) - \frac{(\sigma_y - \sigma_x)}{2}\sin 2\theta \qquad [8]$$

where σ_θ and τ_θ are normal and shear stresses acting on a particular plane and θ is the inclination angle of the plane with the horizontal direction.

If the density of the 'searching grid' is sufficiently fine, it can be assumed that stresses are constant within a 'grid element'. These constant stresses can be signified by stresses at the middle point of the 'grid element'. Therefore, the resistance and shear forces acting on i^{th} segment can be calculated as:

$$R_i = \tau_{f_i} . \Delta L_i = \sum_{ij=1}^{ne} R_{ij} = \sum_{ij=1}^{ne} \tau_{f_{ij}} . l_{ij} \qquad [9]$$

$$R_i = \sum_{ij=1}^{ne} \{c'_{ij} + (\sigma_{ij} - u_a).\tan\phi'_{ij} +$$

$$+ (u_a - u_w).\tan\phi^b_{ij}\}.l_{ij} \quad [10]$$

$$S_i = \tau_i.\Delta L_i = \sum_{ij=1}^{ne} S_{ij} = \sum_{ij=1}^{ne} \tau_{ij}.l_{ij} \quad [11]$$

$$G_i = (R_i - F_s.S_i) \quad [12]$$

where (*ij*) is the 'grid element' passed by the segment $\overline{jk}$, τ_{fij} and τ_{ij} are the shear strength and shear stress mobilized at the middle point of the element (*ij*), c'_{ij}, ϕ'_{ij} and ϕ^b_{ij} are strength parameters of the soil at element (*ij*), *ne* is the number of (*ij*) and l_{ij} is the length of the segment limited by the boundary of (*ij*). The terms τ_{fij}, τ_{ij} and l_{ij} can be called 'element strength', 'element shear stress' and 'element length', respectively (Fig. 3).

Let us introduce an 'optimal function' obtained at stage [*i*], $H_i(j)$, which is equal to the minimum value of the auxiliary function G between the initial stage and point {*j*} in stage [*i*]. According to 'the principle of optimality' (Bellman 1957), the optimal function $H_{i+1}(k)$ obtained at stage [i+1] can be calculated as:

$$H_{i+1}(k) = H_i(j) + G_i(\overline{j,k}) \quad [13]$$

where $H_{i+1}(k)$ is the minimum value of 'auxiliary function', G, between the initial stage and point {*k*} in stage [*i+1*] and $G_i(\overline{j,k})$ is the 'return function' calculated when the optimal search passes from point {*j*} in stage [*i*] to point {*k*} in stage [*i+1*]. At the initial stage, the value of the 'optimal function' $H_1(j)$ is equal to zero, that is:

$$H_1(j) = 0 \qquad j = 1...NP_1 \quad [14]$$

where NP_1 is the number of state points in the initial stage. At the final stage (*i=n+1*), the 'optimal function' $H_{n+1}(k)$ is equal to the minimum value of the 'auxiliary function' G, that is:

$$H_{n+1}(k) = H_n(j) + G_n(\overline{j,k}) \quad [15]$$

$$H_{n+1}(k) = G_m = \sum_{i=1}^{n} (R_i - F_s.S_i) \; ; k = 1...NP_{n+1} \quad [16]$$

where NP_{n+1} is the number of state points in the final stage. The 'optimal point' in the final stage is defined as the point at which, the 'optimal function' calculated is minimum. From the optimal point {*k*} in the final stage, the 'optimal point' {*j*} in the previous stage is also determined. The 'optimal path' defined by connecting the 'optimal points' is eventually found by tracing back from the current stage to the previous stage. Essentially, the 'optimal path' found by tracing back in the field of finite element stresses is the 'critical slip path' through the slope.

It is important to note that the value of F_s in [3] has not been defined in advance. Therefore, the first value of F_s must be assumed prior to the first performance of dynamic programming search. The trial value of F_s is updated using the value of F_s evaluated after each trial of the optimal process. The process will stop as long as the convergence is reached.

(3) Finite element stress analysis and stress interpolation

There are numerous computer softwares developed for stress-strain analysis using finite element method. Recently, the capability of a partial general differential equation solver called FlexPDE (copyright © by PDE Solutions Inc., 1995-2001) for solving geotechnical problems has been well studied by several researchers at the University of Saskatchewan (Vu, 1998; Nguyen, 1998; Pentland, 2000). The software is an improvement of the previous version known as PDEase™ (PDEase is the trademark of Macsyma, Inc.). FlexPDE is a flexible software system for obtaining numerical solutions to single or coupled sets of partial differential equations. FlexPDE allows the user to pose his problem in compact problem-oriented form and proceed directly to graphic presentation of solutions, without digressing to learn the complexities of programming of finite element implementation (http://www.pdesolutions.com).

One of the features of FlexPDE that was useful for the study is that stresses calculated can be output to a designated grid using the finite element function. Stresses stored at Gaussian points are used as the source for interpolation. Therefore, it is of interest to design an output grid in FlexPDE that is coincident with the 'searching grid' in dynamic programming search. In doing so, the stage-state point system in dynamic programming is no longer manually set up. Moreover, the density of the 'searching grid' can also be changed easily by a simple command. As long as the output grid is designed, FlexPDE will perform an interpolation using stresses stored at the Gaussian points and write interpolated stresses to the nodes of the designed grid.

The stress at the middle point of the rectangle 'grid element' is interpolated using the shape function. Suppose that stresses at four nodes of a rectangle element are calculated, the stress at the middle point of the element is:

$$\bar{\sigma} = \sum_{i=1}^{4} N_i . \sigma_i \qquad [17]$$

where $\bar{\sigma}$ is the interpolated stress at the middle point, N_i and σ_i are the shape function and nodal stress. The stress interpolation should be done prior to the performance of the optimal search. Stresses interpolated at the middle of the 'grid element' are used as input data for the dynamic programming search.

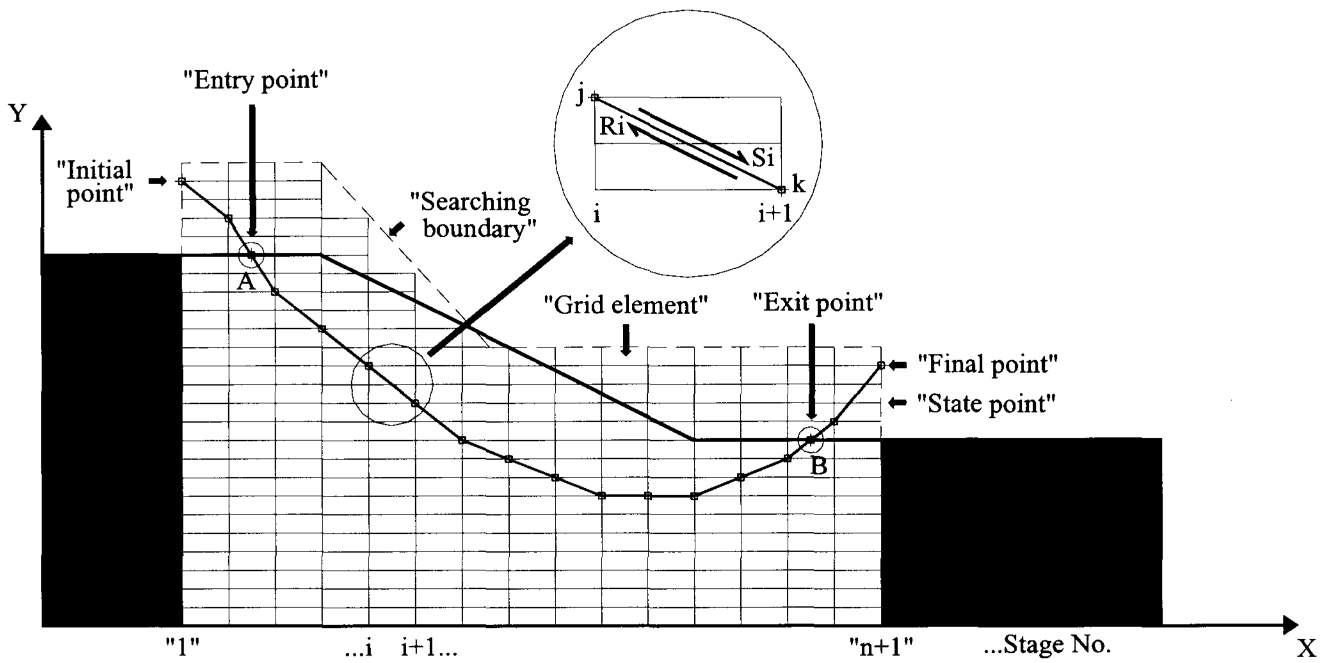

Figure 4. The analytical scheme of slope stability analysis using dynamic programming

(4) Dynamic programming in slope stability analysis

The general theory of dynamic programming can readily be applied to the analysis of slope stability. Figure 4 shows a typical soil slope with stresses analyzed using FlexPDE. Stresses are interpolated and written by FlexPDE to all the nodes of the output grid. The searching grid designed must overlap the physical boundary of the slope but intersect the boundary at one point to prevent the search from choosing the 'optimal path' well outside the slope. The stage-state point system taking the shape of a 'searching grid' is coincident with the output grid of FlexPDE. The boundary of the 'searching grid' is called the 'searching boundary' and should be defined as input data for the optimal search. Since the entry point of the critical slip path is not known, the dynamic programming search should be started from somewhere outside the physical boundary of the slope where stresses are set to zero. Similarly, the dynamic programming search should end at a point outside the slope with zero-stress state. The actual 'entry' and 'exit' point are eventually found at the intersection of the 'optimal path' with the physical boundary of the slope. The procedure for dynamic programming search is as follows:

1. Input the geometry data and soil properties of the problem
2. Import nodal stresses and the output grid from FlexPDE.

3. Set up the searching boundary. Determine the coordinates of state points.
4. Interpolate stresses at the middle point of each 'grid element'.
5. Set a zero-stress state to all state points that outside the physical boundary of the slope.
6. Assume an initial F_s (i.e. F_s=1).
7. Begin the search at all state points located in the initial stage.
8. Calculate values of the 'optimal function' at all points located in the second stage using equations [15], [16] and the assumed F_s.
9. Record the minimum value of the 'optimal function' at each state point in the second stage and the corresponding state point in the previous stage.
10. Proceed to the next stage with the same routine until the final stage is reached.
11. Compare the value of the 'optimal function' obtained at each state point in the final stage. Determined the state point in the final stage at which the value of the 'optimal function' is minimum. The determined state point is the first point of the 'optimal path'.
12. Trace back to the previous stage to find out the corresponding point with the first optimal point in the final stage. The corresponding point is also an optimal point.
13. Keep tracing back to the initial stage to determine the whole 'optimal path'.
14. Evaluate the new F_s of the 'optimal path' using [2]. The newly trial value of F_s is: $F_s^n = \frac{F_s^p + F_s^e}{2}$ where: F_s^n, F_s^p and F_s^e are the newly trial F_s, previously trial F_s and currently evaluated F_s, respectively.
15. The procedure is repeated until the convergence criterion: $\delta \leq \left| F_s^e - F_s^p \right|$ is obtained.
16. Intersections of the 'optimal path' found with the physical boundary of the slope are 'entry' and 'exit' points. The actual critical slip surface is defined as a part of the 'optimal path' from the 'entry' point to the 'exit' point.

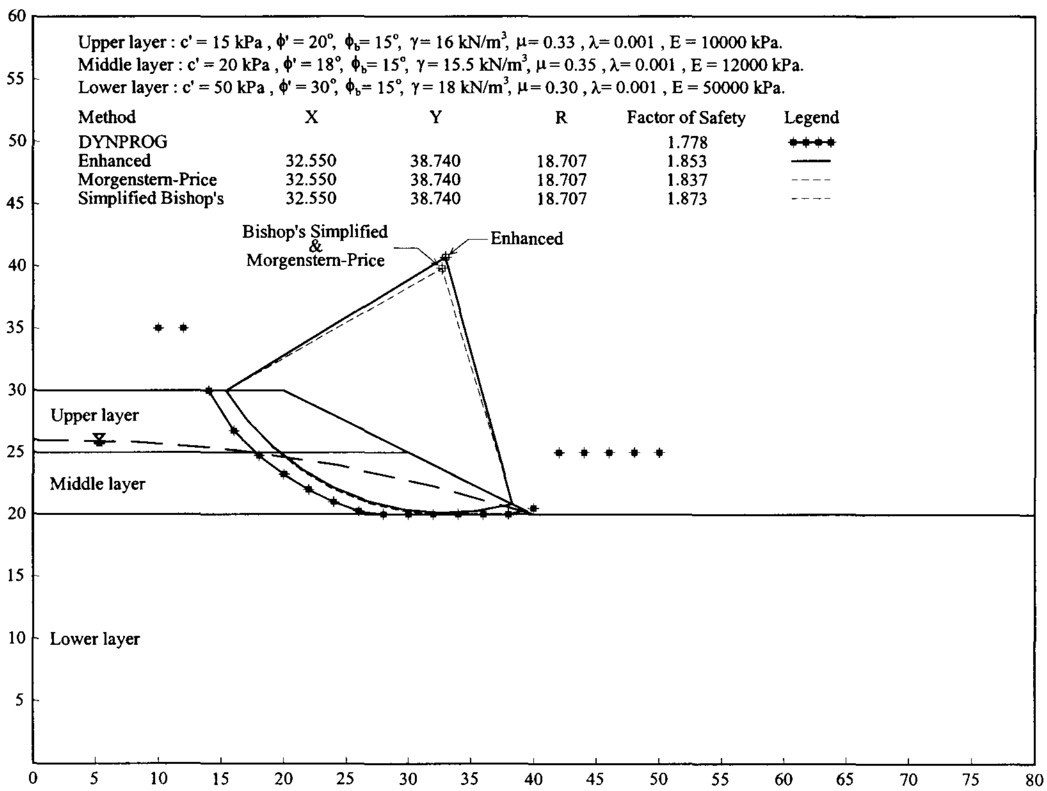

Figure 5. Results of the Example No.1.

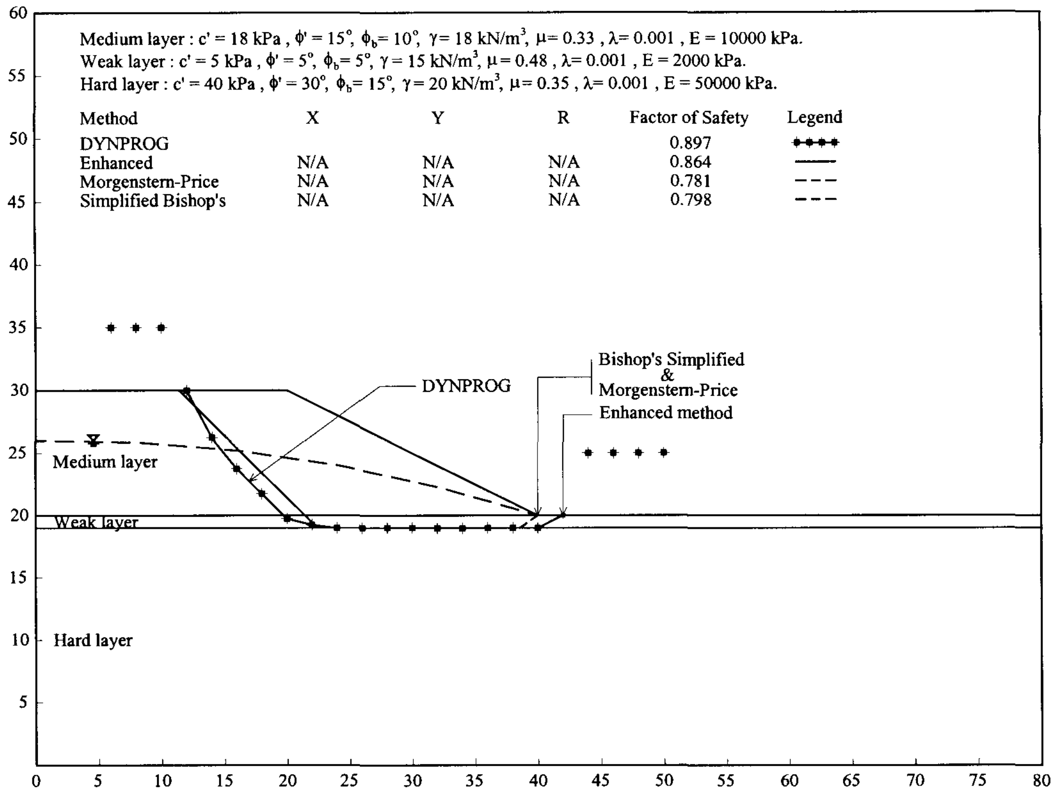

Figure 6. Results of the Example No.2.

3. ILLUSTRATIONAL EXAMPLES

Two examples have been analyzed using a computer program called DYNPROG. The program is developed using the algorithm of dynamic programming coupled with finite element stress analysis. The stress distribution is imported from FlexPDE. The geometry data, soil properties and the 'searching boundary' are defined as the input data for the program. The first example is a multilayer slope resting on a hard foundation. The second example is the analysis of a slope with weak layer sandwiched by two intermediate layers. Each soil is assumed to be homogeneous, isotropic, linear elastic stress-strain behaviour. The comparison has been made to some conventional limit equilibrium methods of slices (i.e., Bishop's Simplified, 1955 and Morgenstern-Price, 1965).

An analysis using the so-called 'enhanced method' has also been made. Essentially, the enhanced method is a limit equilibrium method with normal and shear stress acting at the base of slices evaluated using finite element stress analysis. The enhanced method was introduced by Kulhawy (1969) and recently studied by Fredlund and Scoular (1999). Figure 5 and 6 show results of two examples in which the location and factor of safety of the critical slip surface determined by the proposed method are fairly close to those found by Bishop's Simplified method, Morgenstern-Price method as well as the enhanced method. In the first example, the factor of safety by dynamic programming method is the lowest among the others. In the second example, the 'entry' and 'exit' points found by the proposed method are coincident with those of the enhanced method. However, there is a significant difference in location of the upper part of the critical slip surfaces. The critical slip surfaces found in both examples contain a linear segment coincident with the boundary of the hard layer. The movement of the soil mass indicated in the second example is also reasonable for this particular situation.

4. CONCLUSIONS

A method for slope stability analysis using the dynamic programming technique coupled with a finite element stress analysis is presented. Two examples have been analyzed using the proposed method. The results show that the proposed method gives satisfactory solutions in terms of the location of the slip surface and the value of the factor of safety as compared with those obtained by some conventional methods of slices. The proposed method has some distinguished advantages as compared the conventional limit equilibrium methods. The advantages of the proposed method are:

i) There is no assumption regarding the shape and location of the trial slip surface except the one by which the critical slip path is an assemblage of linear segments. If the 'searching grid' is sufficiently fine, the roughness of the critical slip path can be smoothed.

ii) Since the constitutive laws of soils are preserved by finite element stress analysis, conditions of force and moment equilibrium are rigorously satisfied.

iii) By using the finite element stress analysis, the more sophisticated stress-strain behaviour of soils such as non-linearity can be readily modelled. Therefore, effects of the stress history and Poisson's ratio on the stability of the slope can be studied. These effects are certainly unknown through the use the conventional limit equilibrium methods.

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