Prediction of heave using a general partial differential equation solver

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ABSTRACT: This paper presents a method for prediction of the heave based on the general volume change theory for unsaturated soils. The proposed method makes use of the general purpose partial differential equation solver, PDEase2D, along with non-linear elastic modulus functions. The elastic modulus functions are calculated from conventional oedometer test results. Typical heave problems and case histories are analyzed using the proposed method. The solutions agree well with the analytical solutions and measured results. General partial differential equation solver shows excellent promise for solving volume change problems involving highly non-linear unsaturated soil properties.

1 INRODUCTION

The volume change of an expansive soil, due to the increase of water can distress to engineered structures. Numerous one-dimensional methods have been proposed to predict heave of expansive soils (Fredlund and Rahardjo, 1993). These methods generally make use of a linear relationship between void ratio (or vertical strain) and the log of matric suction (or total stress) in predicting heave. Fredlund et al. (1980) presented a method for the prediction of one-dimensional heave based on the volume change theory of unsaturated soils. The volume change behavior of an unsaturated soil can be discussed in terms of two independent stress state variables; namely the net normal stress $(\sigma - u_a)$ and matric suction $(u_a - u_w)$, where σ is total normal stress, u_a is pore-air pressure and u_w is pore-water pressure. It is assumed that the air phase is continuous and at atmospheric pressure.

This paper proposes an extension of the Fredlund et al. (1980) method for the prediction of heave to two-dimensions. The proposed method makes use of the finite element method, along with non-linear elastic modulus functions. The soil is assumed to be isotropic, non-linear and elastic. The elastic modulus functions (i.e., elastic modulus with respect to net normal stress, E, and elastic modulus with respect to matric suction, H) are computed from oedometer test results (i.e., volume change indices).

The theoretical formulation of the heave prediction method is based on equilibrium equations for the soil structure and the constitutive equations for the unsaturated soil. The solution of a volume change problem is obtained through the use of a general-purpose, partial differential equation solver (i.e., PDEase2D).

2 HEAVE FORMULATION

The force equilibrium of a soil element for K_0-loading condition (i.e., volume change in the y-direction only) can be expressed in the following form:

$$\frac{\partial \sigma_y}{\partial y} + \rho g = 0 \tag{1}$$

where ρ = the density of the soil, g = acceleration due to gravity.

The soil structure constitutive relation for K_0-loading condition can be written as follows (Fredlund and Rahardjo, 1993):

$$(\sigma_y - u_a) = \frac{E(1-\mu)}{(1+\mu)(1-2\mu)}\left(\varepsilon_v - \frac{(1+\mu)}{H(1-\mu)}(u_a - u_w)\right) \tag{2}$$

where E = elastic modulus for the soil structure with respect to net normal stress; H = elastic modulus for the soil structure with respect to matric suction; and μ = Poisson's ratio.

The differential expression of normal strain in terms of displacement in the y-direction (i.e., v) is as follows:

$$\varepsilon_y = \frac{\partial v}{\partial y} \tag{3}$$

where v = vertical displacement.

The governing differential equation for the K_0-loading condition can be obtained by substituting expression for net normal stress (Eq. 2) into the force equilibrium equation (i.e., Eq. 1), assuming the pore-air pressure is atmospheric.

$$\frac{\partial}{\partial y}\left(\frac{E(1-\mu)}{(1+\mu)(1-2\mu)}\left(\frac{\partial v}{\partial y} - \frac{(1+\mu)}{H(1-\mu)}(u_a - u_w)\right)\right) + \rho g = 0 \tag{4}$$

The elastic moduli E and H can be calculated from swelling indices, initial void ratio and assumed Poisson's ratio (Hung, 2000; Hung and Fredlund, 2000) as follow:

$$E = n_t(\sigma - u_a)_{ave} \tag{5}$$

$$H = n_m(u_a - u_w)_{ave} \tag{6}$$

where n_t = the coefficient that relates net normal stress with elastic modulus E; n_m = coefficient that relate matric suction with elastic modulus H; $(\sigma - u_a)_{ave}$ = average of the initial and final net normal stress for an increment; and $(u_a - u_w)_{ave}$ = average of the initial and final matric suction for an increment.

$$n_t = \frac{(1+\mu)(1-2\mu)}{(1-\mu)}\frac{(1+e_0)}{0.4343C_t} \tag{7}$$

$$n_m = \frac{(1+\mu)}{(1-\mu)}\frac{(1+e_0)}{0.4343C_m} \tag{8}$$

where C_t and C_m = volume change indices with respect to net normal stress and matric suction, respectively; and e_0 = initial void ratio.

Equation 4 can be used to compute the vertical displacement due to net normal stress changes and/or matric suction changes.

3 GENERAL-PURPOSE PARTIAL DIFFERENTIAL EQUATION SOLVER

A number of mathematical computer programs have emerged over the last two decades for the solution of math related problems. PDEase2D is one of the general-purpose partial differential equation solvers, and is marketed by Macsyma Inc. The focus of the software package is on ensuring proper converging solutions where highly non-linear equations are involved.

The user of the general partial differential equation software must specify the partial differential equation to be solved. The equation contains the variables to be solved along with a series of material properties. The boundary conditions around a designated region must also be specified.

Equation 4 is non-linear since the elastic moduli are functions of the stress state variables. An incremental procedure is used within PDEase2D to solve these equations (Desai and Christian, 1977), as shown in Figure 1. The total load (or suction change) is divided into increments and one load (or suction) incremental step is applied at a time. During each incremental step, the elastic moduli are assumed to be unchanged. New moduli are selected at the beginning of each new incremental step dependent upon the stress level, and the displacements from each incremental step are accumulated to give to total displacements.

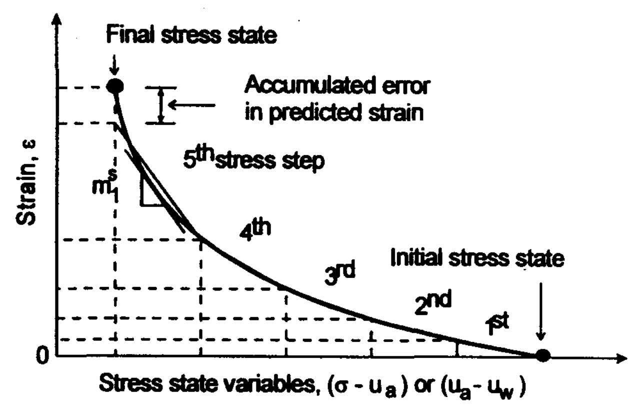

Figure 1. Incremental procedure for performing stress-deformation analysis

4 EXAMPLE PROBLEMS AND CASE HISTORIES

Several typical heave problems and case histories presented by Fredlund and Rahardjo (1993) have been analysed using the proposed method for the heave prediction (Hung, 2000). The predicted results from the proposed method are then compared to existing solutions.

The first example is associated with the total heave of an expansive soil profile below an impermeable cover. The soil below the cover undergoes an increase in pore-water pressure with time. The second example is associated with the heave of an expansive soil profile, when a part of the expansive soil profile is excavated and replaced with an inert granular material.

The first case history is associated with the monitoring of movements of a floor slab in a light industrial building in north central Regina, Saskatchewan, Canada (Yoshida et al, 1983). The second case history is associated with the heave problems in the Eston area of Saskatchewan, Canada (Ching and Fredlund, 1984). There are also measured results that are used to verify the PDEase2D solutions.

The analysis of the first example problem associated with heave due to the increase of pore-water pressure below an impermeable cover is described in Figure 2. The ground surface of an expansive soil is covered with an impermeable layer such as asphalt. With time, the negative pore-water pressure in the soil below the asphalt will increase as a result of the discontinuance of evaporation and evapotranspiration. The heave associated with changes in negative pore-water pressure (i.e., matric suction) is computed.

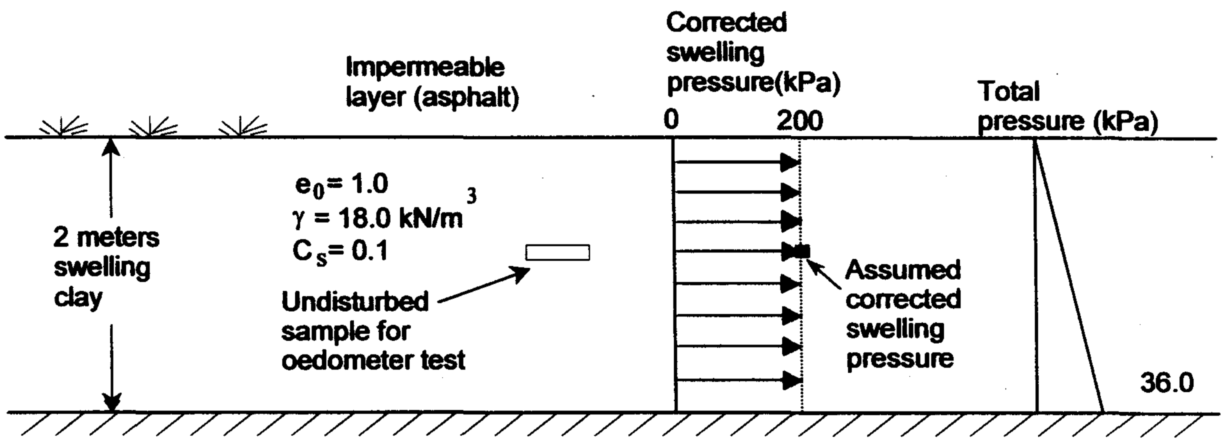

Figure 2. Problem definition for the first example

The final pore-water pressure is assumed to be zero throughout the entire depth. Soil properties and dimensions are shown on Figure 2.

In the Fredlund et al. (1980) method, the heave analysis can be performed by subdividing the active depth into a number of layers. The amount of heave in each layer is computed by considering the stress state changes at the middle of the layer. Figure 3 presents the results of the sensitivity analysis using Fredlund et al. (1980) method. It can be seen that the computed total heave does not change significantly when using more than 25 layers.

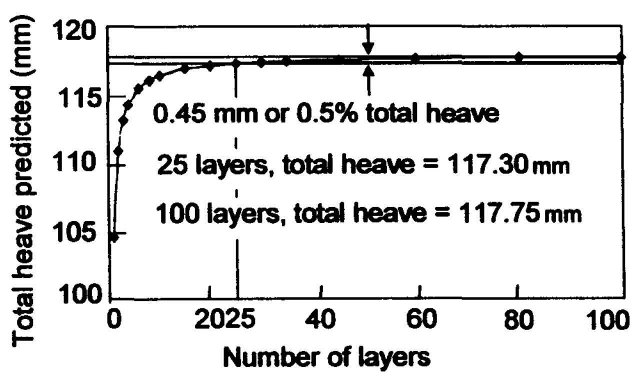

Figure 3. Total heave calculated using Fredlund et al. (1980) method versus number of layers of soils

For the prediction of heave using the finite element method, the elastic modulus function must be evaluated. The matric suction equivalent is used instead of the *in situ* suction. As a result, the measured swelling index on the net normal stress plane, C_{ts}, is used instead of the swelling index with respect to matric suction, C_{ms}. The calculation of elastic modulus was presented in Hung and Fredlund (2000).

For the calculation of the elastic modulus function, the average of the initial and final stress state for an increment must be used in place of the average of matric suction. With an initial void ratio equal to 1.0, swelling index, C_s, equal to 0.1 and assumed Poisson's ratio equal to 0.3, Equation 6 can be rewritten as follows:

$$H = 85.52(u_a - u_w)_{ave} \text{ or } H = 85.52(ST)_{ave} \quad (9)$$

where ST = stress state accounting for overburden pressure and matric suction equivalent.

The stress state is expressed as the sum of overburden pressure and the matric suction equivalent. The elastic modulus function, H, is presented in Figure 4, showing that a constant swelling index on a semi-log plot providing a linear elastic modulus function.

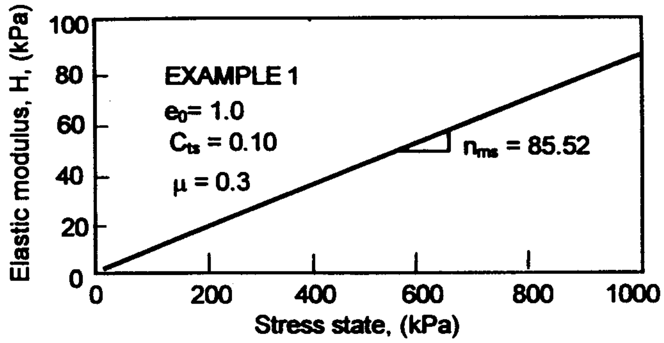

Figure 4. Elastic modulus function, H

The stress-deformation analysis is performed using an incremental procedure with the stress states subdivided into 25 steps. Figure 5 illustrates the initial and final stress state associated with the 1st, 10th, 15th and 25th stress step when using a total of 25 stress steps in the analysis. Figure 5 also shows that the elastic modulus, *H*, decreased with a decreasing stress level near to the ground surface. This results in a greater heave at low stress levels, near to the ground surface.

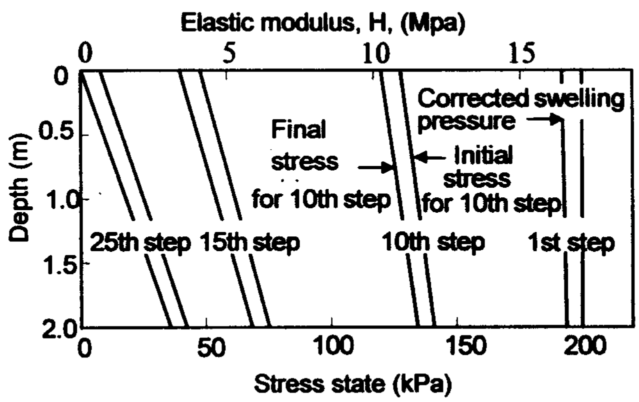

Figure 5. Stress state versus depth

Figure 6 presents the cumulative strain versus depth at various stress steps. More strain is obtained at later stress steps, near to the ground surface.

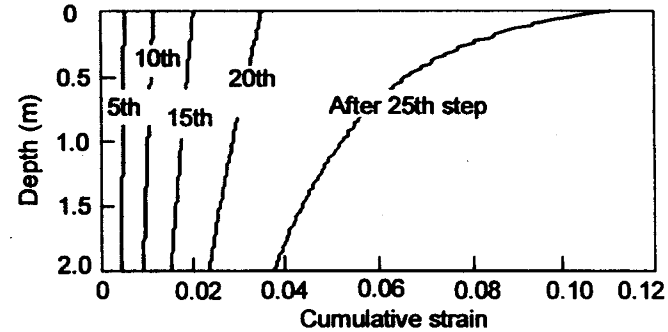

Figure 6. Cumulative strain versus depth at different stress steps

Figure 7 presents the results of the sensitivity analysis for the problem using the PDEase2D program. The figure shows that the total heave calculated increases almost linearly with decreasing the size of the stress step. The total heave predicted when using a stress step size of 1 kPa (200 stress steps) is 117 mm while the total heave predicted when using a size of stress step of 7.2 kPa (25 stress steps) is 115 mm. With small stress step such as 1 kPa, the error is about 0.5% from the correct solution (Fig. 7). The variation in the number of stress increments used from 25 increments to 200 increments results in a change of about 2% in the calculated total heave, for this example. The smaller the size of a stress increment, the more stress steps that are needed. However, it is not practical to use extremely fine size stress steps to perform an analysis. It can be seen that, with a limited number of stress steps, the method would under-estimate the total heave (e.g., by about 3% the correct amount of heave).

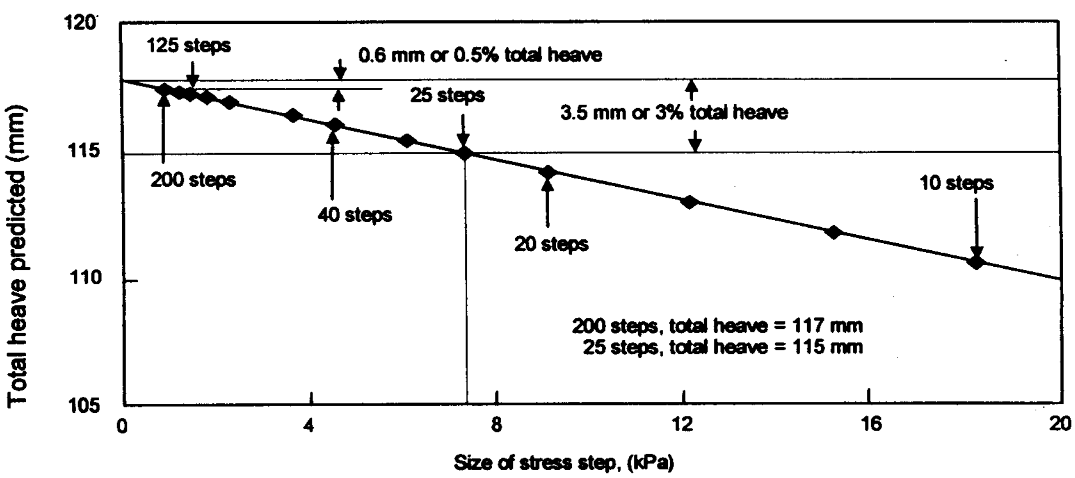

Figure 7. Total heave versus size of stress step for the first example

The values of accumulated heave obtained from the analytical method, as well as from PDEase2D program, are presented in Figure 8. Greater amounts of heave are predicted at later time steps, near to the ground surface where the elastic modulus, H, is small. The amount of heave that PDEase2D predicted when using 25 stress steps is 115 mm. The solution of PDEase2D matches well with that of the Fredlund et al. (1980) method.

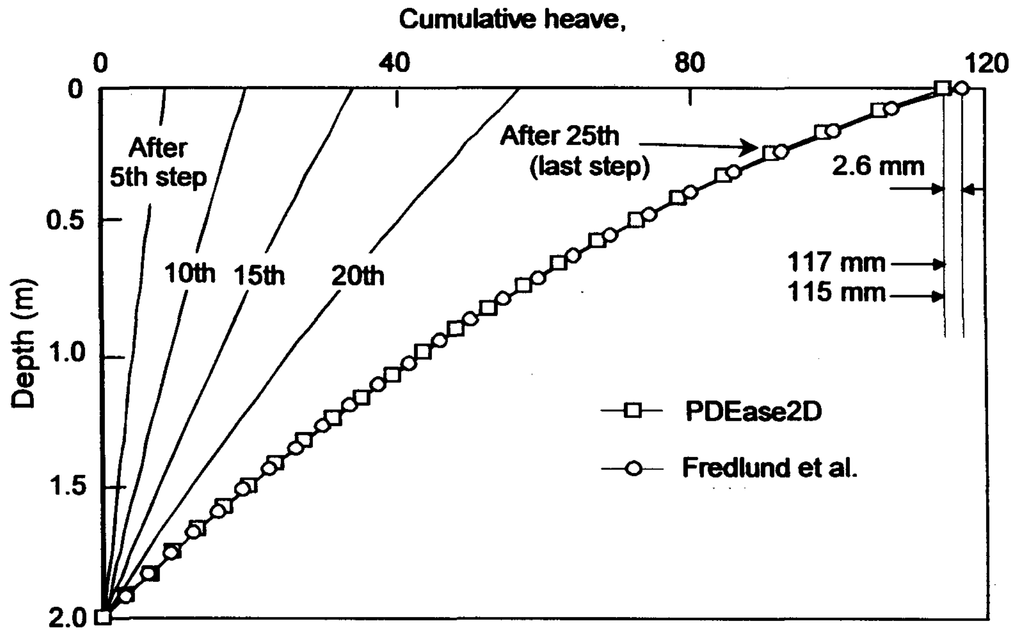

Figure 8. Comparison of the total heaves predicted for the first example

The results of analysis and measured data for the case history associated with Regina clay are presented in Figure 9. The calculated results agree well with the measured data. Figure 10 presents the comparison of the total heave predicted for all heave problems considered in this study. Figure 10 shows that in all cases, the heave predicted using the finite element method agrees well with that obtained from the Fredlund et al. (1980) method.

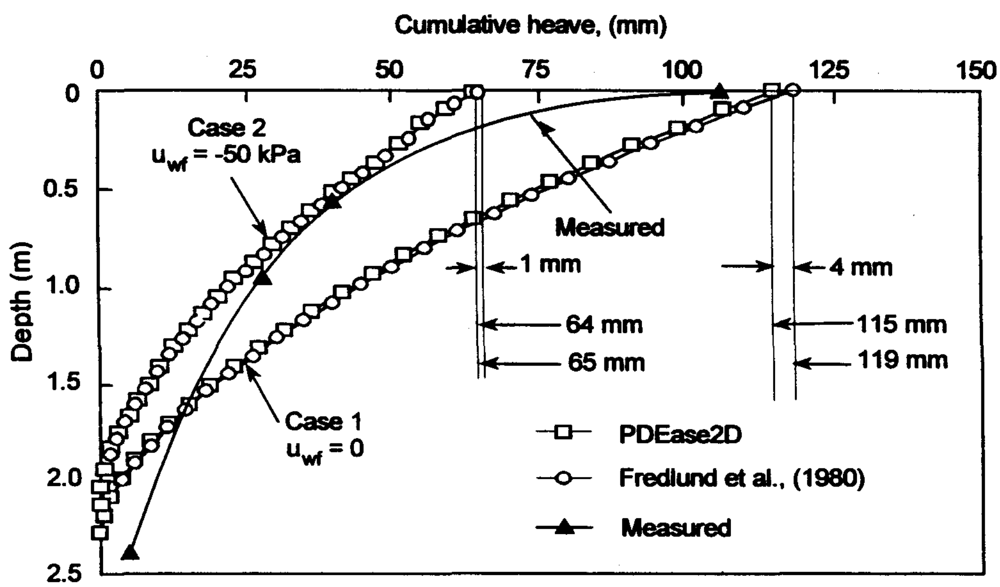

Figure 9. Comparison of the total heaves predicted and measured data for the first case history

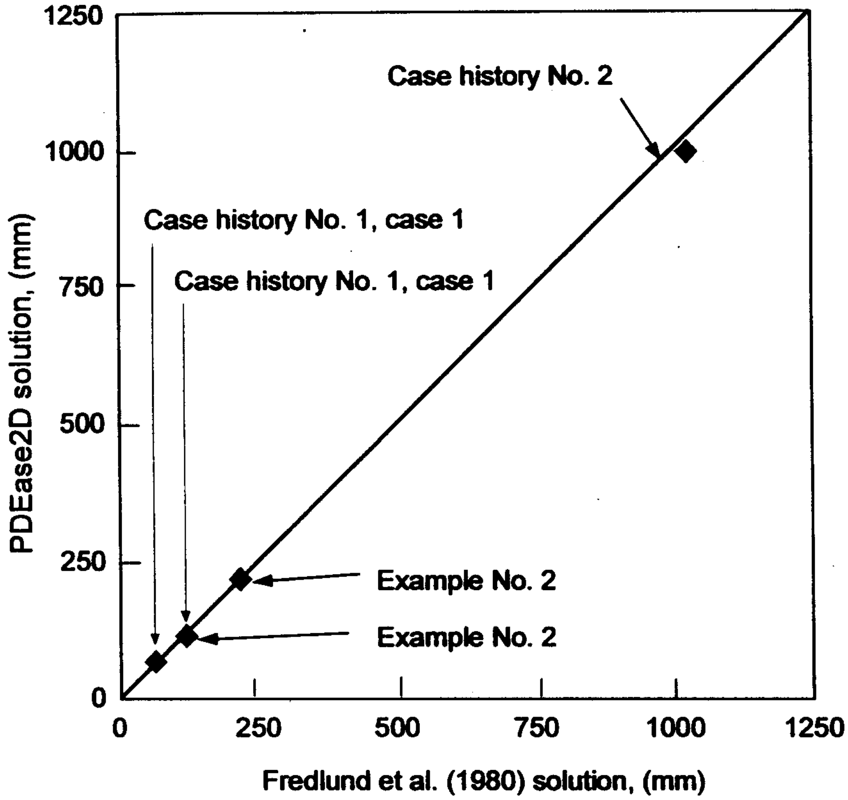

Figure 10. Comparison between the total heave predicted for all examples and case histories

5 CONCLUSION

It can be concluded that elastic modulus functions for an expansive soil can be calculated from one-dimensional oedometer test results and used to predict one-dimensional heave with the use of PDEase2D finite element solver. Comparison of the total heave predicted for all heave problems considered in this study shows that the heave predicted using the finite element method agrees well with that obtained from the Fredlund et al. (1980) method and the measured data. General-purpose partial differential equation solver such as PDEase2D shows excellent promise for solving heave problems with highly non-linear unsaturated soil properties.

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