Volume change predictions in expansive soils using a two-dimensional finite element method

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ABSTRACT: The solution of a two-dimensional volume change problem associated with an unsaturated, expansive soil is proposed in a manner consistent with the theory of unsaturated soil behavior. The elastic modulus functions must be computed from conventional laboratory test results. Procedures for the calculation of elastic modulus functions associated with an unsaturated soil are presented. Predictions of two-dimensional heave are performed in an uncoupled manner through the use of a general-purpose, partial differential equation solver, called PDEase2D. An example problem involving volume change due to changes in matric suction is used to illustrate the proposed method of volume change prediction and the use of the PDEase2D computer program.

1 INTRODUCTION

Volume change in an unsaturated soil is caused by changes in either or both the stress state variables, net normal stress and matric suction. The volume change and water flow processes are dependent processes, requiring a coupled analysis. The soil properties associated with an unsaturated soil are dependent on stress state variables, and the governing partial differential equations become highly non-linear. Therefore, a coupled solution is difficult to obtain. An approximate solution can be obtained by performing the analysis in an uncoupled manner, where the continuity and equilibrium equations are solved independently.

This paper presents an uncoupled solution of volume change problems associated with an expansive soil under plane strain loading conditions. The governing partial differential equations for seepage and stress-deformation are based on the general theory of unsaturated soil behavior. The air phase is assumed to be continuous and at atmospheric pressure. The net normal stress is therefore equal to net total stress and the matric suction is equal to the absolute value of the pore-water pressure. The soil is assumed to be isotropic, non-linear and elastic. It is also assumed that the elastic modulus with respect to net normal stress, E, obtained from the net normal stress plane, and the elastic modulus with respect to matric suction, H, obtained from matric suction plane can be applied to the entire constitutive surface.

Theory associated with saturated-unsaturated seepage is presented in Thieu et al. (2000), and is not repeated in this paper. Solutions of non-linear partial differential equations for both saturated-unsaturated seepage and stress-deformation are obtained using PDEase2D computer program.

2 VOLUME CHANGE THEORY

Two independent stress state variables are needed to describe the volume change behavior of an unsaturated soil (Fredlund and Morgenstern, 1977). The two stress state variables are net normal stress, $(\sigma - u_a)$, and matric suction, $(u_a - u_w)$, where σ is total normal stress, u_a is pore-air pressure and u_w is pore-water pressure.

Assuming the soil behaves in an incrementally isotropic, linear elastic material, the soil structure constitutive relations can be written as follows:

$$d\varepsilon_{ij} = \frac{1+\mu}{E} d(\sigma_{ij} - u_a) - \frac{\mu}{E} d(\sigma_{kk} - 3u_a)\delta_{ij} + \frac{d(u_a - u_w)}{H}\delta_{ij} \quad (1)$$

where ε_{ij} = components of the strain tensor for the soil structure, δ_{ij} = the Kronecker delta, μ = Poisson's ratio, E = modulus of elasticity for the soil structure with respect to a change in net normal stress, and H = modulus of elasticity for the soil structure with respect to a change in matric suction.

The total volumetric deformation of an unsaturated soil element, $d\varepsilon_v$, can be written as the sum of the normal strains:

$$d\varepsilon_v = \frac{dV_v}{V_0} = d\varepsilon_x + d\varepsilon_y + d\varepsilon_z \tag{2}$$

where: $d\varepsilon_x$, $d\varepsilon_y$, $d\varepsilon_z$ = normal strain components in x, y, and z-direction, respectively.

2.1 *Calculation of elastic moduli from volume change indices.*

Fredlund and Rahardjo (1993) presented the relationship between the volumetric strain change, $d\varepsilon_v$, and changes in stress state variables in an elasticity form for various loading conditions. These constitutive equations are based on Eqs. (1) and (2), and can be written in a compressibility form as follow:

$$d\varepsilon_v = m_1^s d(\sigma - u_a) + m_2^s d(u_a - u_w) \tag{3}$$

where: $m^s{}_1$ = coefficient of volume change with respect to a change in net normal stress, and $m^s{}_2$ = coefficient of volume change with respect to a change in matric suction.

The relationships between the coefficients of volume change and the elastic moduli are presented in Table 1 for general, three-dimensional loading, and plane strain loading conditions.

The constitutive relationship for the soil structure of an unsaturated soil can be presented graphically as a three-dimensional surface (Fig. 1). Coefficients of volume change corresponding to the unloading surface can be subscripted with an "s" to represent the word swelling (i.e., $m^s{}_{1s}$ and $m^s{}_{2s}$).

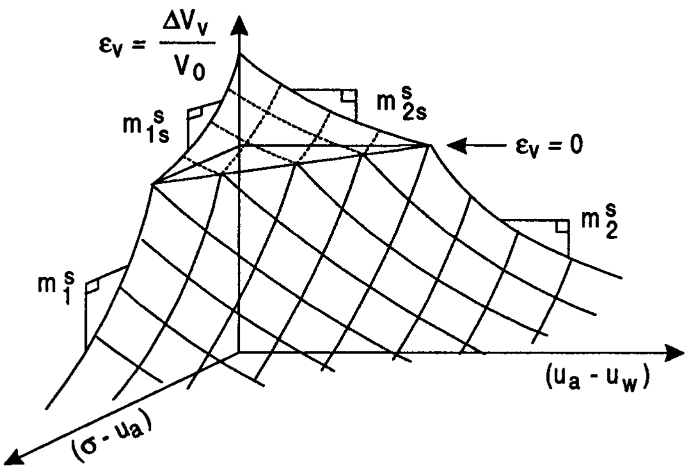

Figure 1. Three-dimensional constitutive surface for soil structure of an unsaturated soil (Fredlund and Rahardjo, 1993)

The constitutive surface can also be obtained when void ratio, e, is plotted with respect to the logarithms of the stress state variables (Fig. 2). The logarithmic plots are essentially linear over a relatively large stress range on the extreme plans (i.e., the $\{log(\sigma\text{-}u_a)\approx 0\}$ plane and $\{log(u_a\text{-}u_w)\approx 0\}$ plane) (Fredlund and Rahardjo, 1993). The volume change indices for the unloading surface are subscripted with an "s" as C_{ts} and C_{ms}.

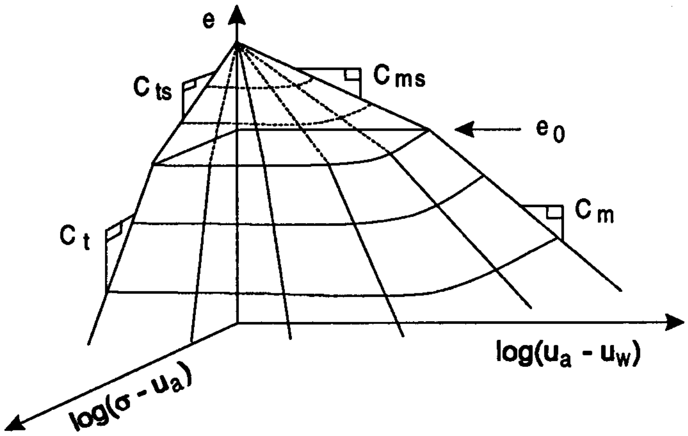

Figure 2. Semi-logarithmic plot of void ratio versus stress state variables (Fredlund and Rahardjo, 1993)

Using a mathematical conversion between a semi-logarithmic scale and arithmetic scale, the coefficients of volume change can be written in term of the volume change indices:

$$m_1^s = \frac{0.434 C_t}{1+e_0} \frac{1}{(\sigma - u_a)_{ave}} \tag{4}$$

$$m_2^s = \frac{0.434 C_t}{1+e_0} \frac{1}{(u_a - u_w)_{ave}} \tag{5}$$

where: $(\sigma - u_a)_{ave}$ = average of the initial and final net normal stress for an increment, $(u_a - u_w)_{ave}$ = average of the initial and final matric suction for an increment.

The elastic moduli, E and H, can be calculated from the volume change indices, initial void ratio and Poisson's ratio by substituting Eqs. (4) and (5) into conversion relationships shown in Table 1.

$$E = n_t(\sigma - u_a)_{ave} \tag{6}$$

$$H = n_m(u_a - u_w)_{ave} \tag{7}$$

where: n_t = coefficient that relates net normal stress with elastic modulus E, and n_m = coefficient that relate matric suction with elastic modulus H.

Table 2 presents the n_t and n_m coefficients for general, three-dimensional loading, and plane strain loading conditions. Figure 3 illustrates the relationship between elastic modulus, E, and net normal stress for various value of swelling index, when the initial void ratio is equal to 1.0 and Poisson's ratio equal to 0.35. The relationship between elastic modulus, H, and matric suction, is plotted for various values of swelling index in Fig. 4.

2.2 *Stress-deformation formulation*

Equations of equilibrium for the soil structure of an unsaturated soil are:

$$\sigma_{ij,j} + b_i = 0 \tag{8}$$

Table 1 Coefficient of volume change for some loading conditions (From Fredlund and Rahardjo, 1993)

Loading	m_1^s	First stress state variable	m_2^s	Second stress state variable
Three-dimensional (General)	$3\dfrac{(1-2\mu)}{E}$	$d(\sigma_{mean}-u_a)$	$\dfrac{3}{H}$	$d(u_a-u_w)$
Plane strain (Two-dimensional)	$\dfrac{2(1+\mu)(1-2\mu)}{E}$	$d(\sigma_{ave}-u_a)$	$\dfrac{2(1+\mu)}{H}$	$d(u_a-u_w)$

Table 2 Coefficients n_t and n_m for some loading conditions

Loading	n_t	First stress state variable	n_m	Second stress state variable
Three-dimensional (General)	$\dfrac{6.908(1-2\mu)(1+e_0)}{C_t}$	$(\sigma_{mean}-u_a)_{ave}$	$\dfrac{6.908(1+e_0)}{C_m}$	$(u_a-u_w)_{ave}$
Plane strain (Two-dimensional)	$\dfrac{4.605(1+\mu)(1-2\mu)(1+e_0)}{C_t}$	$(\sigma_{ave}-u_a)_{ave}$	$\dfrac{4.605(1+\mu)(1+e_0)}{C_m}$	$(u_a-u_w)_{ave}$

Note: $\sigma_{mean} = (\sigma_x+\sigma_y+\sigma_z)/3$; $\sigma_{ave} = (\sigma_x+\sigma_y)/2$

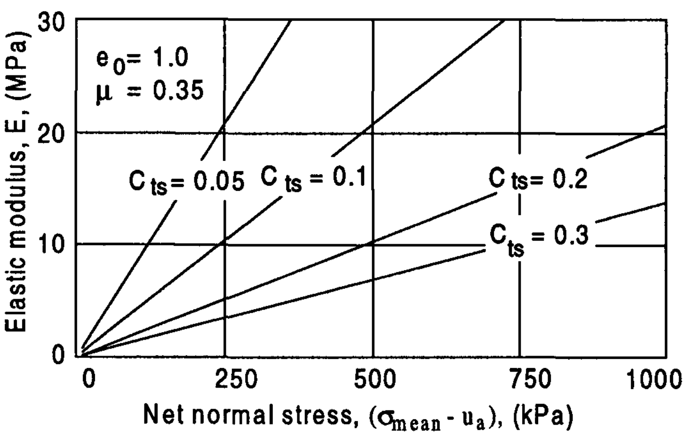

Figure 3. Relationship between elastic modulus, E, and net normal stress for various values of swelling index

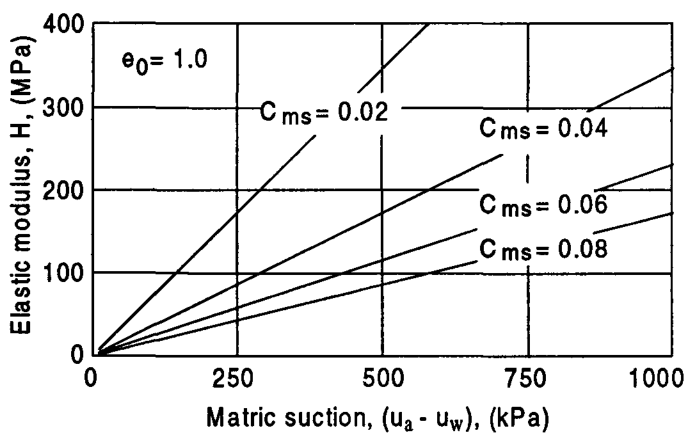

Figure 4. Relationship between elastic modulus, H, and matric suction for various values of swelling index

where σ_{ij} = components of the net total stress tensor, and b_i = components of the body force vector.

The partial differential equations for the soil structure can be derived from constitutive equations (Eq. (1)) and the force equilibrium equations (Eq. (8)). The partial differential equations in term of displacements in x and y-direction (i.e., u and v) for plane strain loading ($d\varepsilon_z = 0$) are as follows:

Equations 9 and 10 are non-linear since the elastic moduli are functions of the stress state variables. An incremental procedure can be used to solve these equations (Desai and Christian, 1977). The incremental procedure for solution is illustrated in Fig. 5. The total load (or suction change) is divided into increments and one load (or suction) incremental step is applied at a time. During each incremental step,

$$\frac{\partial}{\partial x}\left\{c\left[(1-\mu)\frac{\partial u}{\partial x}+\mu\frac{\partial v}{\partial y}-\frac{(1+\mu)}{H}(u_a-u_w)\right]\right\}+\frac{\partial}{\partial y}\left\{G\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)\right\}=0 \tag{9}$$

$$\frac{\partial}{\partial x}\left\{G\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)\right\}+\frac{\partial}{\partial y}\left\{c\left[\mu\frac{\partial u}{\partial x}+(1-\mu)\frac{\partial v}{\partial y}-\frac{(1+\mu)}{H}(u_a-u_w)\right]\right\}+\rho g=0 \tag{10}$$

where $c=\dfrac{E}{(1-2\mu)(1+\mu)}$; $G=\dfrac{E}{2(1+\mu)}$; ρ = density of the soil, g = acceleration due to gravity the elastic moduli are assumed to be unchanged. New moduli are selected at the beginning of the new incremental step based on the stress level, and the displacements from each incremental step are accumulated to give to total displacements. The computer program, PDEase2D, was used to perform the stress-deformation analysis.

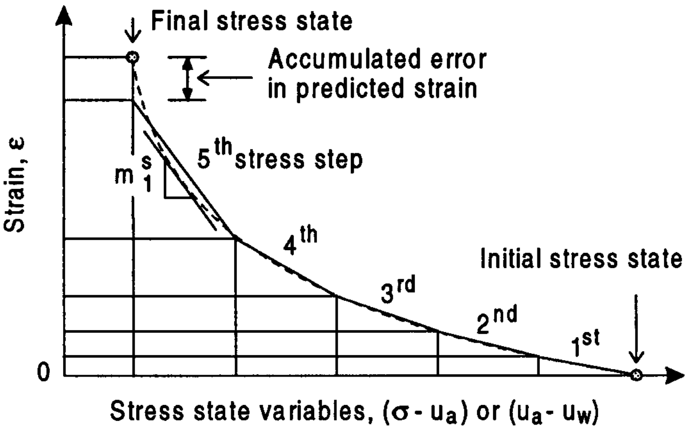

Figure 5. Incremental procedure for stress-deformation analysis

3 EXAMPLE PROBLEM AND COMPUTER RESULTS

A typical volume change example was analyzed to demonstrate the application of the two-dimensional volume change prediction method (Hung, 2000). The example considers a 5 m thick layer of expansive clay soil (Fig. 6). The coefficient of permeability of the soil was described using Gardner's equation (1958) with a saturated coefficient of permeability equal to $1.16\text{x}10^{-9}$ m/s, a parameter *a* equal to 0.001 and a parameter *n* equal to 2. The initial void ratio of the soil is equal to 1.0, and the swelling index with respect to matric suction, C_{ms} is equal to 0.07.

The initial matric suction was taken to be constant throughout the depth and equivalent to 700 kPa. It was then assumed that a leaking water line produced zero pore-water pressure under the cover. The soil is then watered at the surface with an infiltration rate of $1.16\text{x}10^{-10}$ m/s, The water table is 15 m below the ground surface.

Deformations in the soil profile due to changes in matric suction from initial to final state (i.e., steady state) are predicted.

The elastic modulus function *H* with respect to matric suction for the soil with a given initial void ratio, swelling index, and an assumed Poisson's ratio equal to 0.3 can be calculated from Eq. 7 and Table 2 and written as follows:

$$H = 171.05(u_a - u_w)_{ave} \tag{11}$$

A seepage analysis was performed to predict final matric suction conditions. Boundary conditions and the finite element mesh for the seepage analysis is shown in Fig. 7. Zero flow is specified at the left and the right boundaries. A zero total head is specified at the flexible cover and –15 m total head is specified at the lower boundary. A boundary flow value of $1.16\text{x}10^{-10}$ m/s is specified along the uncovered surface. The maximum error of 0.1% is specified for the problem. The finite element mesh satisfying the specified error has 89 six-node-triangular elements and 206 nodes.

The matric suction distribution in the soil at equilibrium under specified boundary condition is shown in Fig. 8. The suction change below the cover is more then that below the exposed area, and therefore more heaves would be expected at the cover area.

Figure 9 shows the finite element mesh and the boundary conditions specified for the stress-deformation analysis. At the left and right sides of the domain, the soil is free to move in the vertical direction and is fixed in horizontal direction. The lower boundary is fixed in both directions.

The problem is analyzed using twenty-five equal suction increments. The size of stress step varies from 23 kPa to 27 kPa. Maximum error specified for stress-deformation analysis is 1%. The finite element mesh varies with step numbers, from 25 elements and 78 nodes for the first step, to 55 elements and 134 nodes for the last step.

The cumulative heave at surface is shown in Fig. 10 for various stress steps. The total heave at ground

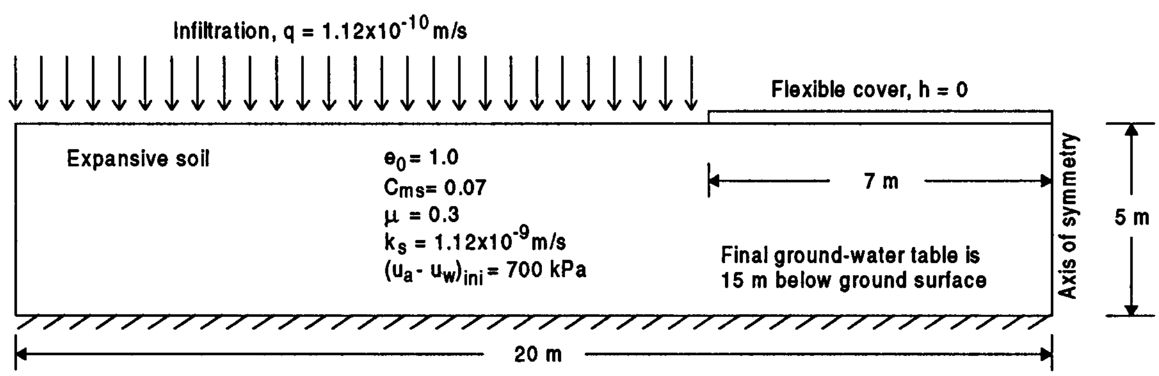

Figure 6. Illustration of an example problem

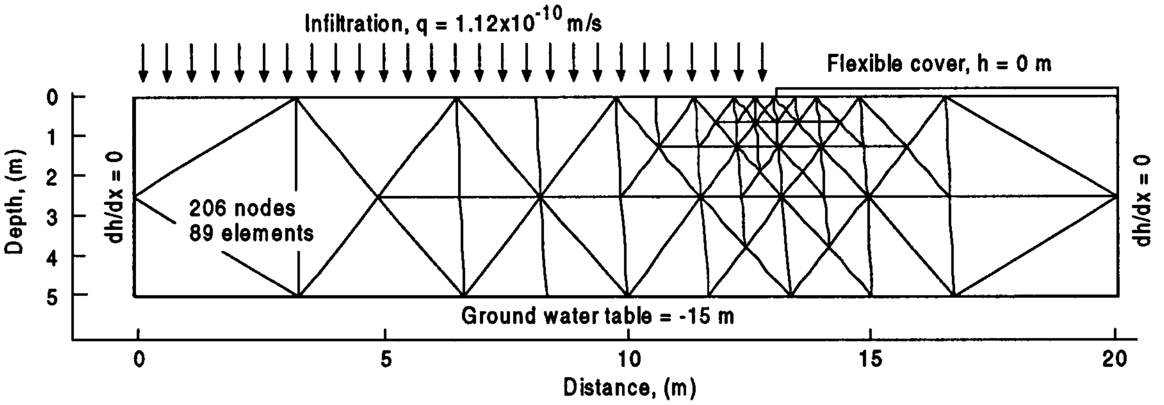

Figure 7. Finite element mesh and boundary condition for unsaturated seepage analysis

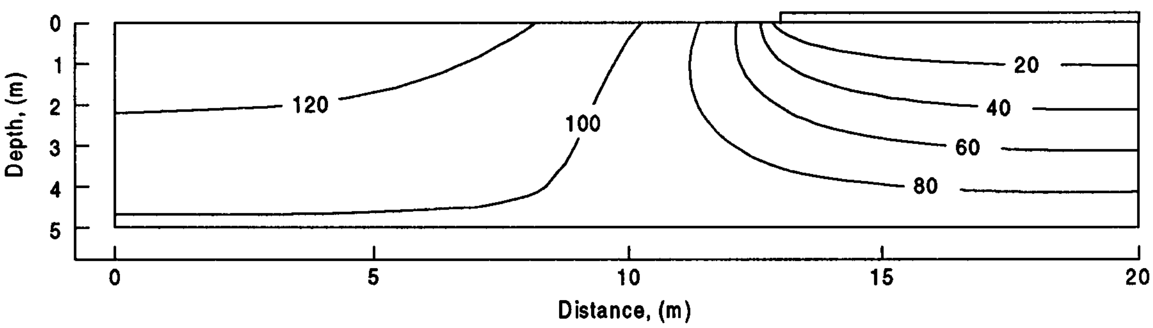

Figure 8. Final matric suction profile (steady-state conditions)

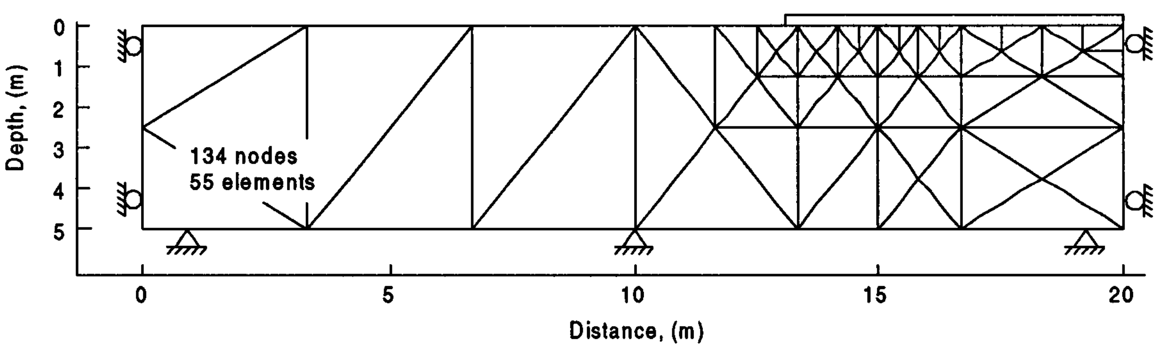

Figure 9. Finite element mesh and boundary conditions for stress-deformation analysis

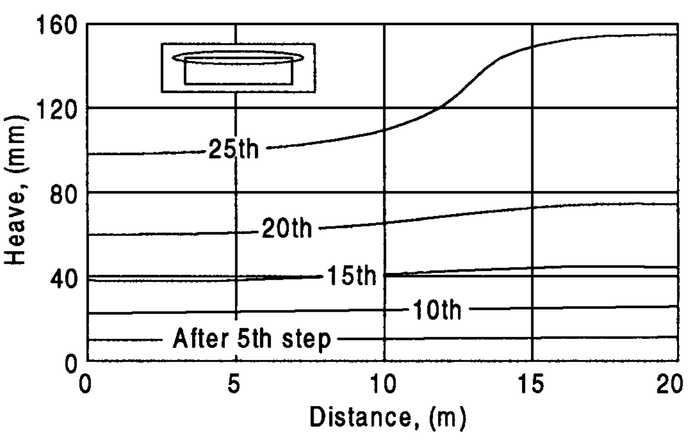

Figure 10. Cumulative heave at surface at various stress steps

surface under the cover is 155 mm, while it is only about 100 mm at the exposed area. The differential heave is 55 mm. Figure 11 presents the cumulative heave below the cover. Figures 10 and 11 show that most of displacements take place during the later stress step, because the value of elastic modulus get lower as matric suction is reduced. Contours of total heaves are presented in Fig. 12 and the distribution of the deformation vectors is shown in Fig. 13.

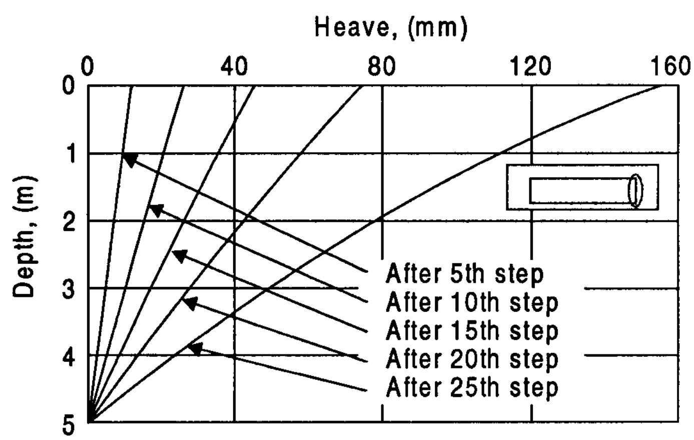

Figure 11. Cumulative heave below the cover (at x = 20 m)

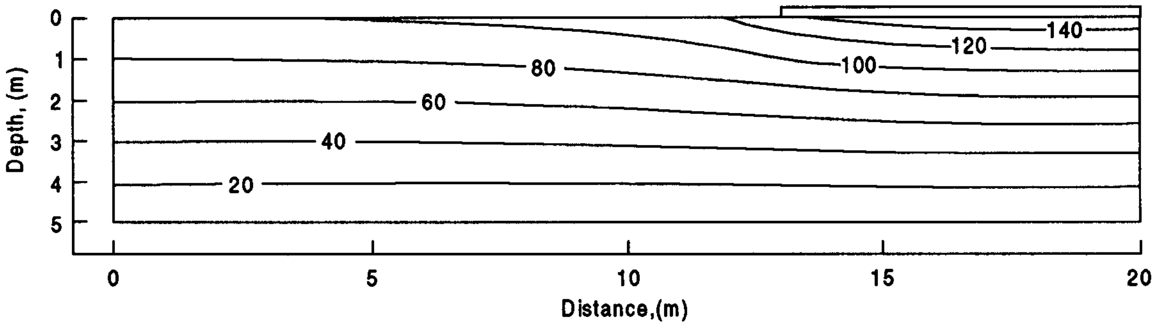

Figure 12. Distribution of total heave contours (mm)

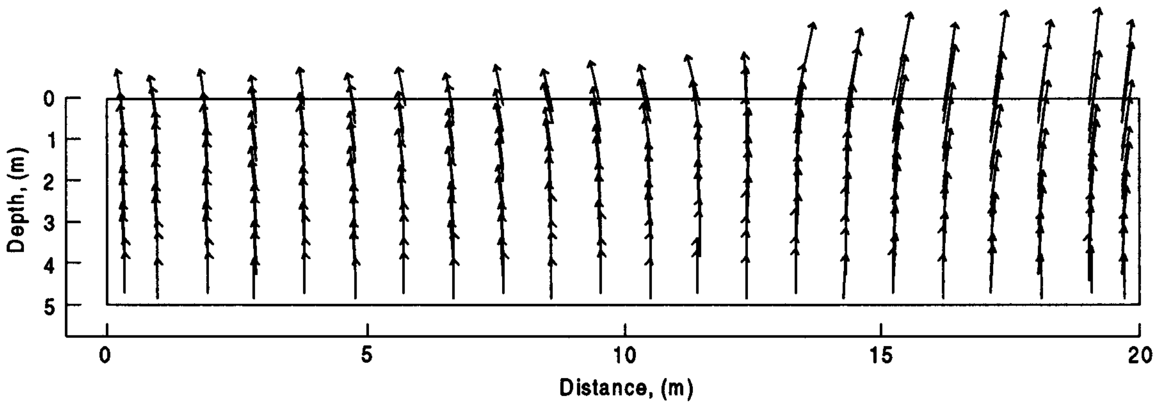

Figure 13. Distribution of heave vectors

4 CONCLUSION

The volume change index with respect to net normal stress in the net normal stress plane and the initial void ratio are required for the calculation of elastic modulus function, E. The volume change index with respect to matric suction in the matric suction plane and the initial void ratio are required for the calculation of elastic modulus function, H. A value for Poisson's ratio must be assumed.

Two-dimensional volume change of an expansive soil has been analyzed in an uncoupled manner using partial differential equation solvers. The matric suction conditions in soils can be predicted by performing a saturated/unsaturated seepage analysis. Stress-deformation analysis then can be performed to predict the deformations.

General-purpose partial differential equation solvers, such as PDEase2D, appear to be a powerful tool for solving seepage and volume change problems associated with an unsaturated soil.

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