Proceedings of the GeoDenver Conference, Denver, Colorado, pp. 69-83, August 3-8, 2000

Use of Grain-Size Functions in Unsaturated Soil Mechanics

Murray D. Fredlund[1]
G. Ward Wilson[2]
Delwyn G. Fredlund[2]

Abstract

The grain-size distribution is commonly used for soil classification; however, there is also potential to use the grain-size distribution as a basis for estimating unsaturated soil behavior. Mathematically representing the grain-size distribution provides several benefits to soil mechanics. An example is the estimation of the soil-water characteristic curve from the grain-size distribution. Much emphasis has recently been placed on the estimation of unsaturated soil-property functions from the soil-water characteristic curve. Several methods have been proposed that use the grain-size distribution as the basic information for the estimation of the soil-water characteristic curve.

Two mathematical forms are presented to represent grain-size distribution curves; namely, a unimodal form and a bimodal form. The equations presented in this paper can provide a close representation of a wide variety of grain-size distribution for different soil types.

1.1 Introduction

The grain-size distribution is a simple, yet informative classification test routinely performed in soil mechanics. Valuable information regarding the amount of each particle size can be determined in the laboratory through the use of a series of sieves and hydrometer analysis. Recent research has made use of the grain-size distribution as the basis for the estimation of soil properties such as the soil-water characteristic curve (Gupta and Larson, 1979; Arya and Paris, 1981; Haverkamp and Parlange, 1986). It is of value to be able to mathematically represent the grain-size distribution curve as a continuous function that will allow further analysis to be performed. The

[1] Graduate student ,Department of Civil Engineering, University of Saskatchewan, Saskatoon, Sask., S7N 5A9
[2] Professor of Civil Engineering, Department of Civil Engineering, University of Saskatchewan, Saskatoon, Sask., S7N 5A9

specific objective of this paper is to develop a mathematical description for the grain-size distribution of any given soil. This forms the foundation step for the general procedure to determine related soil property functions and constitutive relations.

Mathematically representing the grain-size distribution provides several benefits to soil mechanics. Firstly, the soil can be classified using the best-fit parameters. Secondly, the mathematical equation can be used as the basis for further soil analysis and description. An example is the estimation of the soil-water characteristic curve from the grain-size distribution (Fredlund et al, 1997). Thirdly, a mathematical equation can provide a method of representing the entire curve between measured data points.

Two models to fit grain-size data are proposed in this paper. These models consist of a unimodal and a bimodal mathematical function. The two new equations provide great flexibility for fitting a wide variety of soils.

1.2 Definition of variables

The grain-size distribution for a soil is defined as the relationship between percent passing (by mass) and the particle size. It has also been called the mass-based aggregate size distribution or the ASD. The *particle size* represents the size of particles that can pass a particular sieve mesh. The *percent passing* represents the mass percentage of particles passing a particular sieve size.

1.3 Background

ASTM D422-54T (1958) presented a standard for determining the grain-size distribution. Standard sieve sizes, reporting methods, and methods for performing a hydrometer analysis are presented. The sieve analysis allowed points on the grain-size distribution to be determined for particle sizes greater than the #200 sieve or 0.074 mm. The hydrometer analysis presented by ASTM, standardizes a method for determining the grain-size distribution for particles smaller than the #200 sieve. Interpretation of the grain-size distribution is typically carried out manually.

Gardner (1956) used a two-parameter, log-normal distribution to fit grain-size distribution data. Kemper and Chepil (1965) further confirmed the work of Gardner (loc cit.). The log-normal distribution often failed to provide a close fit of the grain-size distribution at the extremes of the curve (Gardner, 1956; Hagen et al, 1987). Wagner and Ding (1994) later improved upon the log-normal equation by presenting three and four parameter log-normal equations.

Campbell (1985) presented a classification diagram based on the assumption that the particle-size distribution is approximately log-normal. This assumption led to the particle-size distribution being approximated with a Gaussian distribution function. With this assumption, any combination of sand, silt, and clay can be represented by a geometric (or log) mean particle diameter and a geometric standard deviation. Values were summarized in a modified USDA textural classification chart by Shirizi and Boersma (1984).

The first limitation associated with using a log-normal type of equation is the assumption that the grain-size distribution is symmetric. In reality, the grain-size distribution is often non-symmetric and can be better fit by a different type of equation. Secondly, a method for fitting soils that are bimodal or gap-graded is often of value and the four-parameter log-normal equations have not been found to be satisfactory for fitting gap-graded grain-size distributions.

There are three primary types of grain-size distributions. These three types of distributions are known as *well-graded* soils, *uniform* soils, and *gap-graded* soils. Figure 1 illustrates each type of grain-size distribution. This paper focuses on these three categories of grain-size distributions and provides equations to fit the experimental data for each category. Well-graded soils and uniform soils are examined and a unimodal method of fitting an equation is developed. Then a mathematical method of representing a gap-graded soil is subsequently presented.

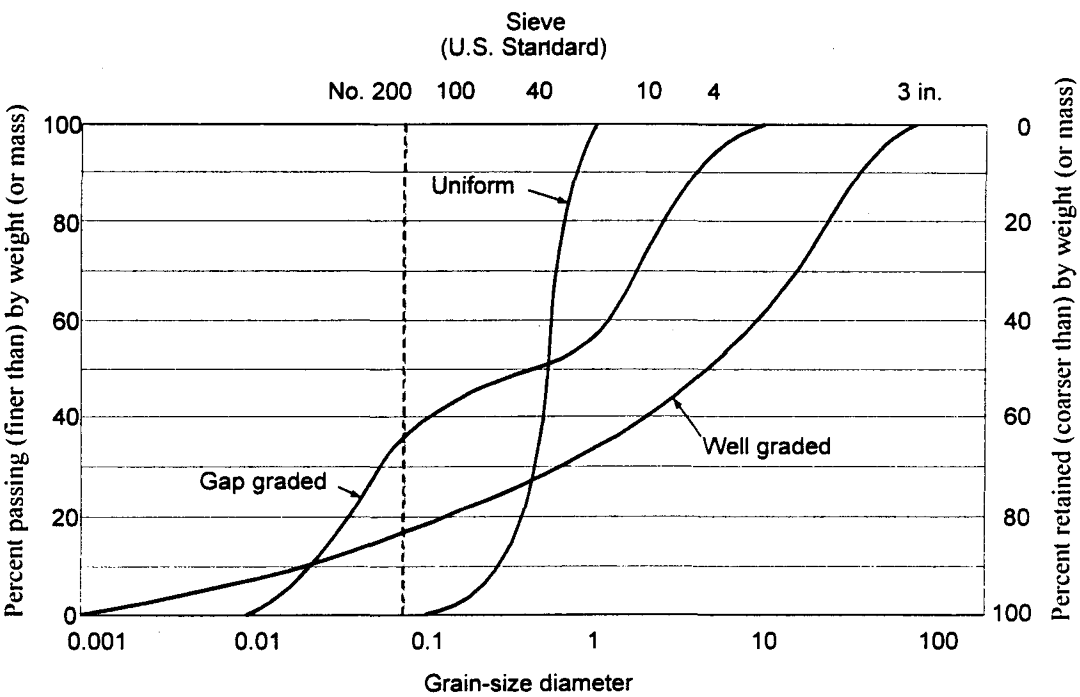

Fig. 1 Three primary types of grain-size distribution curves (Holtz and Kovacs, 1981)

1.4 Unimodal Equation for the Grain-size Distribution

The selection of an appropriate, mathematical equation involves a review of a variety of equations that could be used to fit soils data. It has been observed that the soil-water characteristic curve possessed a shape similar to that of the grain-size distribution. This is to be expected since the soil-water characteristic curve describes the void distribution in a soil while the grain size curve provides information on the distribution of the solid phase of the soil. Since the solids plus the voids add up to the total soil volume, it is to be expected that the distribution of the solids phase (i.e., grain-size distribution) would tend to bear an inverse type relationship to the distribution of voids (i.e., soil-water characteristic curve), and vice versa.

Equations used to fit the soil-water characteristic curve have been proposed by Brooks and Corey, 1964; Gardner, 1974; van Genuchten, 1980; Burdine, 1953; Mualem, 1976; Fredlund and Xing, 1994. Brooks and Corey (1964) and Gardner (1974) presented three parameter equations while van Genuchten (1953) and Fredlund and Xing (1994) presented four parameter equations. It would appear that a similar forms of equations could be used to represent the grain-size distribution.

An accurate representation of the clay fraction of the grain-size distribution was considered necessary in order to complete the mathematical function. The Fredlund and Xing (1994) equation allows independent control over the lower end of the curve (i.e., the fine particle size range), and was selected as the basis for the development of a grain-size distribution equation. The reversed scale of the grain-size distribution as well as characteristics unique to the grain-size distribution, required that the original Fredlund and Xing (1994) equation to be modified to the form shown below:

$$P_p(d) = \frac{1}{ln\left[exp(1) + \left(\frac{a_{gr}}{d}\right)^{n_{gr}}\right]^{m_{gr}}} \left[1 - \left[\frac{ln\left(1 + \frac{d_{rgr}}{d}\right)}{ln\left(1 + \frac{d_{rgr}}{d_m}\right)}\right]^7\right] \quad [1]$$

where: a_{gr} = parameter equal to the inflection point on the curve and related to the initial breaking point of the curve,
n_{gr} = parameter related to the steepest slope of the curve,
m_{gr} = parameter related to the shape of the curve,
d_{rgr} = parameter related to the diameter of the fines in a soil,
d = diameter of any particle size under consideration, and
d_m = diameter of the minimum allowable size particle.

Equation [1] is referred to as the unimodal equation and can be used to fit a wide variety of soils as shown in Figs. 2, and 3. A quasi-Newton least squares regression algorithm was used to adjust three of the five parameters to fit the equation to each soil. The algorithm progressively minimizes the squared differences between the equation and experimental data. The best-fit particle size distribution function can be plotted over the grain-size distribution data, typically on a logarithmic scale.

The unimodal equation provides significant improvements in the fit of grain-size data over previous mathematical representations (i.e., log-normal distribution). The particle size distribution provides information on the amount and dominant sizes of particles present in a soil. Another form can also be used to visualize the distribution of particle sizes by differentiating the particle size distribution curve. The differentiation produces a particle-size probability density function (PDF). The differentiated form of the unimodal grain-size equation can be seen in Eq. [2]. The parameters presented in the particle-size probability density function, PDF, are the same as defined in Eq. [1].

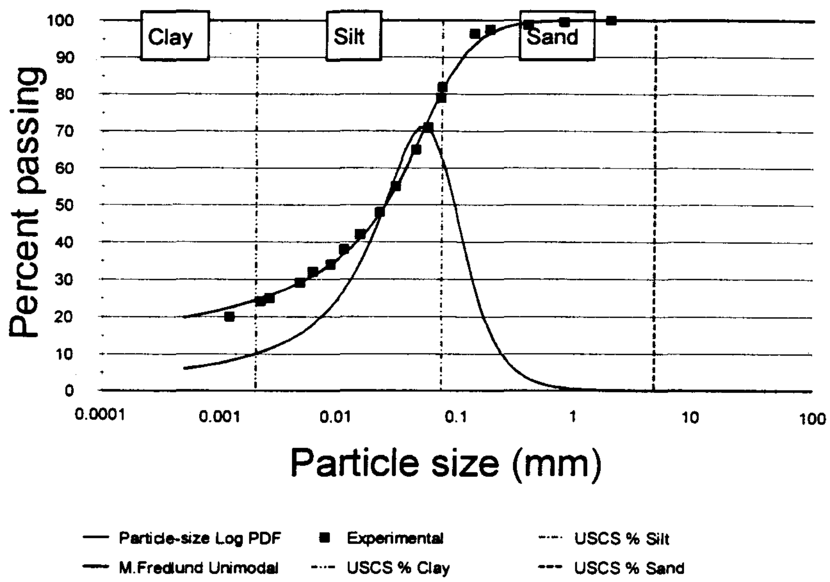

Fig. 2 Logarithmic probability density function for uniform silt from the Pilot Butte area of Saskatchewan (optimum compression) best fit with the unimodal equation.

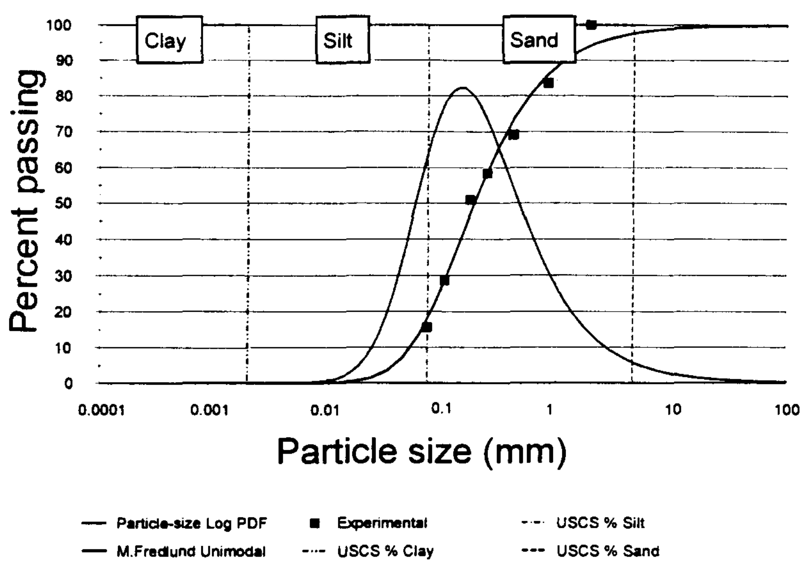

Fig. 3 Logarithmic probability density function for Rubicon Sandy Loam best fit with the unimodal equation.

$$\frac{dp_p}{dd} = \frac{1}{\left[ln\left[exp(1) + \left(\frac{a_{gr}}{d} \right)^{n_{gr}} \right]^{m} \right]} \left[1 - \frac{ln\left(1 + \frac{d_{rgr}}{d} \right)^7}{ln\left(1 + \frac{d_{rgr}}{d_m} \right)^7} \right] m_{gr} \left(\frac{a_{gr}}{d} \right)^{n_{gr}} \frac{n_{gr}}{\left[d\left[\left[exp(1) + \left(\frac{a_{gr}}{d} \right)^{n_{gr}} \right] ln\left[exp(1) + \left(\frac{a_{gr}}{d} \right)^{n_{gr}} \right] \right] \right]}$$

$$+ \frac{7}{ln\left[exp(1) + \left(\frac{a_{gr}}{d} \right)^{n_{gr}} \right]^{m\,gr}} \frac{ln\left(1 + \frac{d_{rgr}}{d} \right)^6}{ln\left(1 + \frac{d_{rgr}}{d_m} \right)^7} \frac{d_{rgr}}{\left[d^2 \left(1 + \frac{d_{rgr}}{d} \right) \right]} \qquad [2]$$

The particle size distributions presented in this paper are calculated using Eq. [2]. The highest point on the PDF plot is the mode or the most frequent particle size. Since Eq. [2] is a PDF, the natural laws of probability hold and the area under the differentiated curve must equal 1 as shown below.

$$\int_{-\infty}^{+\infty} \frac{dP_p}{dd} dx = 1 \qquad [3]$$

where: x = particle-size diameter.

Equation [2] can also be used to calculate probabilities. Equation [4] shows how to calculate the probability that a soil particle diameter will fall in a certain range. Equation [2] can be arithmetically integrated between specified particle diameter sizes and the probability can be determined by the following relationship.

$$\text{probability}(d_1 < d < d_2) = \int_{x=d_1}^{x=d_2} p(x)dx \qquad [4]$$

It is convenient to represent the PDF function in a different manner when plotting on a logarithmic scale. The arithmetic PDF function will often appear distorted when plotted on a logarithmic scale. The peak of Eq. [4] will not represent the most frequent particle size because of the logarithmic distribution of the particle-size scale. To overcome this limitation, the PDF function is often represented as shown in Eq. [5]. Taking the log of particle size and differentiating the grain-size equation produces a PDF that appears more physically realistic. The peak of Eq. [5] will represent the most frequent particle size.

$$p_l(d) = \frac{dP_p}{d\log(d)} = \frac{dP_p}{dd} \ln(10) \cdot d \qquad [5]$$

where: $p_l(d)$ = logarithmic probability density function.

The probability of the logarithmic PDF can be calculated as follows.

$$\text{probability}(d_1 < d < d_2) = \int_{x=\log(d_1)}^{x=\log(d_2)} p_1(x)dx \qquad [6]$$

The probability density function for various grain-size curves can be seen in Figs. 2, and 3.

1.5 Parametric Study of the Proposed Grain-size Distribution Equation

A parametric study of the proposed unimodal equation shows behavior similar to that of the original Fredlund and Xing (1994) equation. The a_{gr} parameter is related to the initial break of the equation and its effect on the grain-size distribution curve can be seen in Fig. 4 where a_{gr} is varied and the other equation parameters are held constant. The a_{gr} parameter provides an indication of the largest particle sizes.

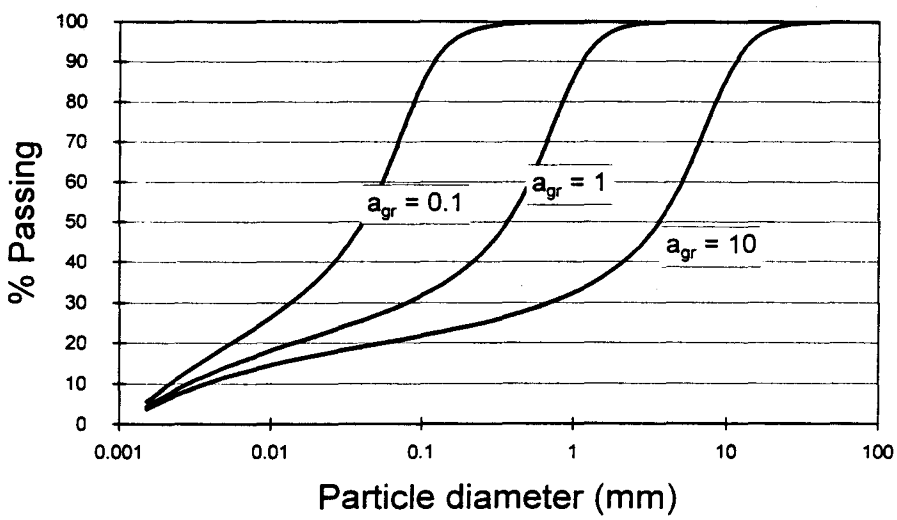

Fig. 4 Effect of varying the a_{gr} parameter while n_{gr} = 4.0, m_{gr} = 0.5, dr_{gr} = 1000, and d_m = 0.001.

Figure 5 shows how the parameter n_{gr} influences the slope of the grain-size distribution. The point of maximum slope of the grain-size distribution indicates the gradation of the particle sizes (i.e., on a logarithm scale) in the soil as seen in Fig. 5. The parameter m_g controls the break onto the finer particle size of the sample. The effect of the m_g parameter can be seen in Fig. 6. The parameter, d_{rgr}, affects the shape of this fine particle size of the curve. However, the amount of variation produced on the curve is quite minimal as shown in Fig. 7. In some cases the d_{rgr} can be modified to improve the fit of the overall equation. It was found that a value of 0.001 for d_{rgr} provided a reasonable fit in most cases.

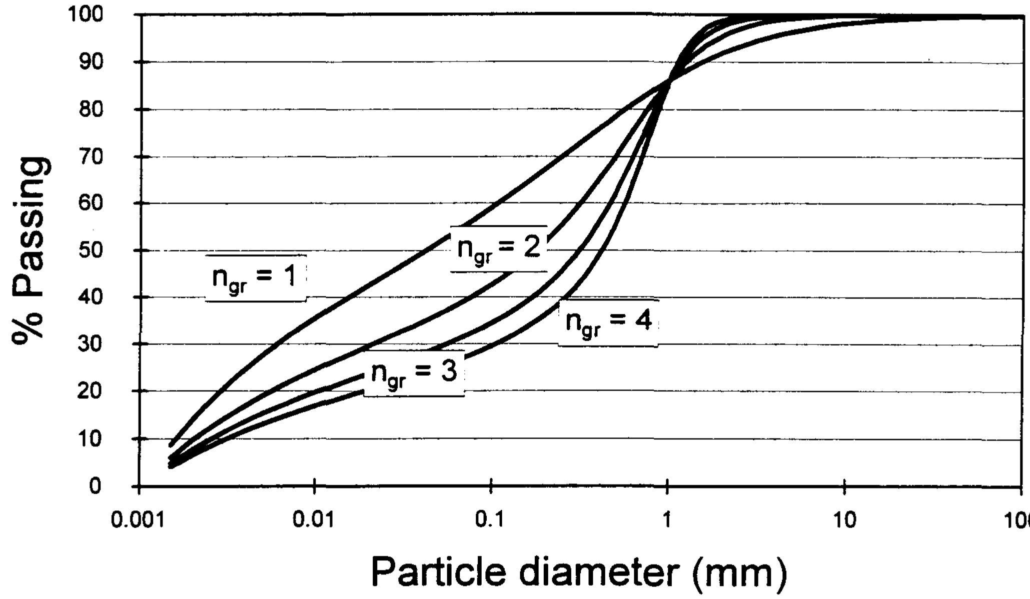

Fig. 5 Effect of varying the n_{gr} parameter while $a_{gr} = 1.0$, $m_{gr} = 0.5$, $dr_{gr} = 1000$, and $dm = 0.001$

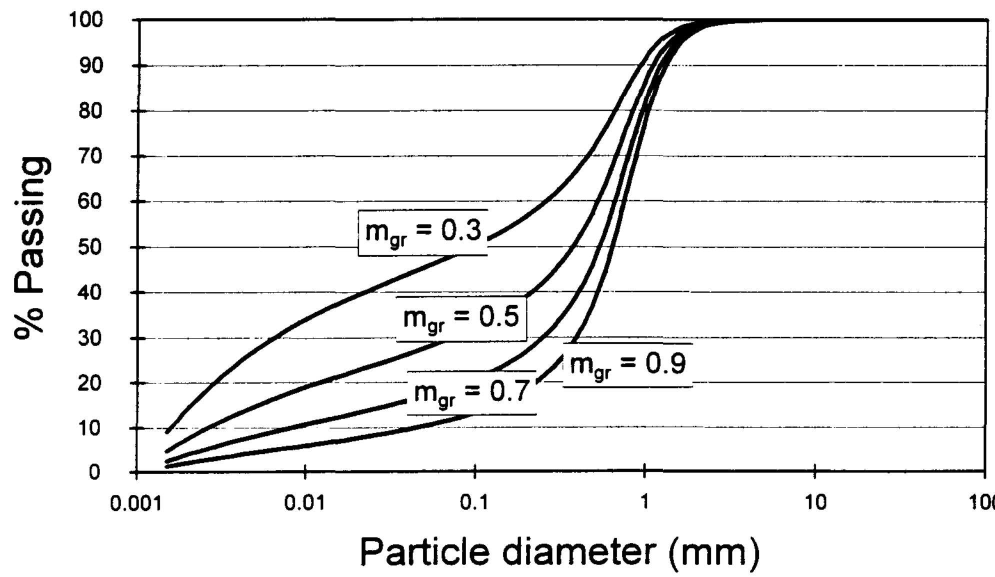

Fig. 6 Effect of varying the m_{gr} parameter while $a_g = 1.0$, $n_{gr} = 4.0$, $dr_{gr} = 1000$, and $d_m = 0.001$

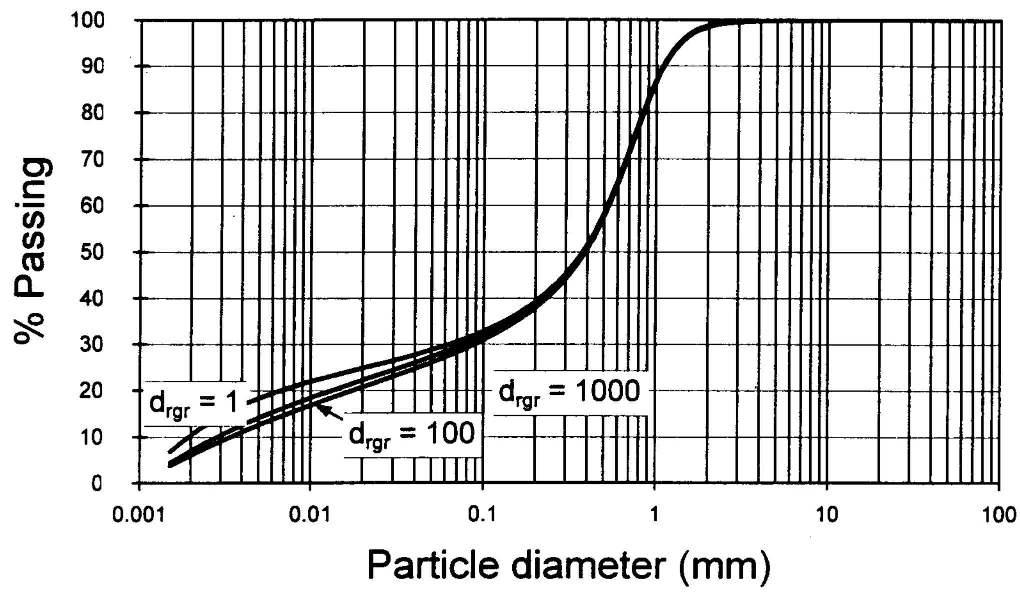

Fig. 7 Effect of varying the d_{rgr} parameter while a_{gr} = 1.0, n_{gr} = 4.0, m_{gr} = 0.5, and d_m = 0.001

The unimodal equation (Eq. [1]) appears to have versatility in handling a wide variety of soil types.

1.6 Bimodal Equation for the Grain-size Distribution Curve

There is a limitation in using the unimodal equation (i.e., Eq. [1]) when the soils are gap-graded. In this case, it is necessary to consider the use of a bimodal, best-fit. Soils frequently have particle size distributions that are not consistent with a unimodal distribution and as a result, attempts to fit the unimodal equation to certain data sets can often lead to unsatisfactory results.

The characteristic shape of a bimodal or gap graded soil is the double "hump" seen in the experimental data. These anomolies indicate that the particles are concentrated around two separate particle size ranges. From a mathematical standpoint, a gap-graded soil can be viewed as a combination of two or more separate soils (Durner, 1994). This allows for the "stacking" of more than one unimodal equation.

$$P_p(d) = \left[\sum_{i=1}^{k} w_i \left[\frac{1}{\ln\left[\exp(1) + \left(\frac{a_{gr}}{d}\right)^{n_{gr}}\right]^{m_{gr}}}\right]\right]\left[1 - \left[\frac{\ln\left(1 + \frac{d_r}{d}\right)}{\ln\left(1 + \frac{d_r}{d_m}\right)}\right]^7\right] \quad [7]$$

where: k = the number of "subsystems" for the total particle-size distribution,
w_i = the weighting factors for the subcurves, subject to $0 < w_i < 1$ and
$\Sigma w_i = 1$.

For a bimodal curve, k would be equal to 2 and the number of parameters to be determined would be 4 times [k + (k-1)]. The unimodal equation is used as the basis for the bimodal equation. The final equation for a bimodal curve is shown below in its extended form.

$$P_p(d) = \left\{ w\left[\frac{1}{\ln\left(\exp(1) + \left(\frac{a_{bi}}{d}\right)^{n_{bi}}\right)^{m_{bi}}}\right] + (1-w)\left[\frac{1}{\ln\left(\exp(1) + \left(\frac{j_{bi}}{d}\right)^{k_{bi}}\right)^{l_{bi}}}\right]\right\}\left[1 - \left(\frac{\ln\left(1 + \frac{dr_{bi}}{d}\right)}{\ln\left(1 + \frac{dr_{bi}}{d_m}\right)}\right)^7\right] \quad [8]$$

where: a_{bi} = parameter related to the initial breaking points on the curve,
n_{bi} = parameter related to the steepest slope on a portion of the curve,
m_{bi} = parameter related to the shape of the curve,
j_{bi} = parameter related to the second breaking point along the curve,
k_{bi} = parameter related to the second steep slope along the curve,
l_{bi} = parameter related to the second shape of the curve,
dr_{bi} = parameter related to the amount of fines in a soil,
d = diameter of any particle size under consideration, and
d_m = diameter of the minimum allowable size particle.

The bimodal data sets can be closely fit using the bimodal best-fit equation (Fig. 8). However, the bimodal fit provided only an *adequate* fit of unimodal data sets. In other words, unimodal data sets were better fit using the unimodal equation. The results of fitting the bimodal curve to several different soils can be seen in Figs. 8 to 10.

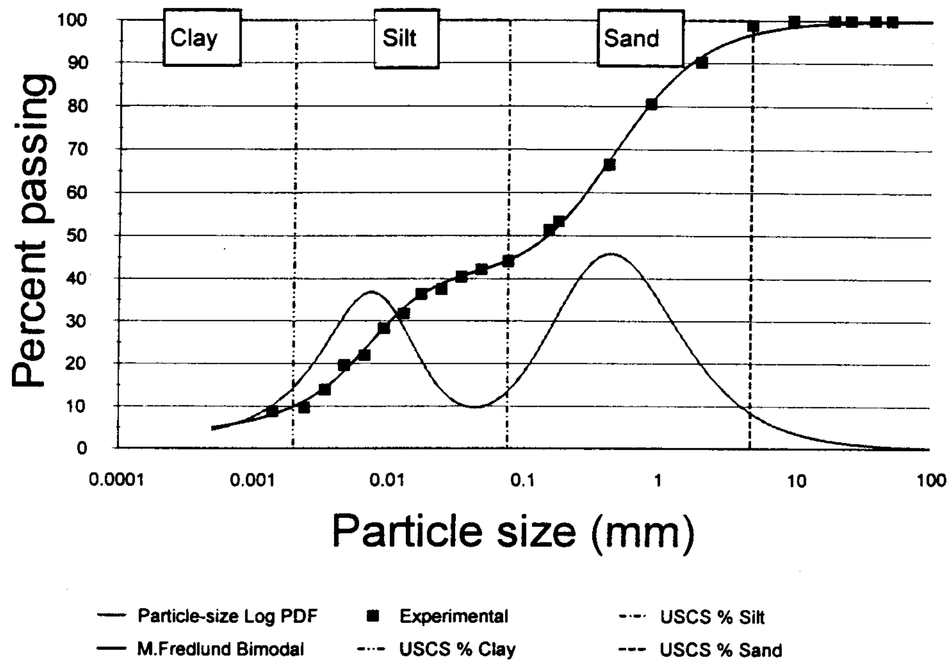

Fig. 8 Logarithmic probability density function fitted with a bimodal equation, for a gap-graded Saprolitic Soil tested at the University of Saskatchewan

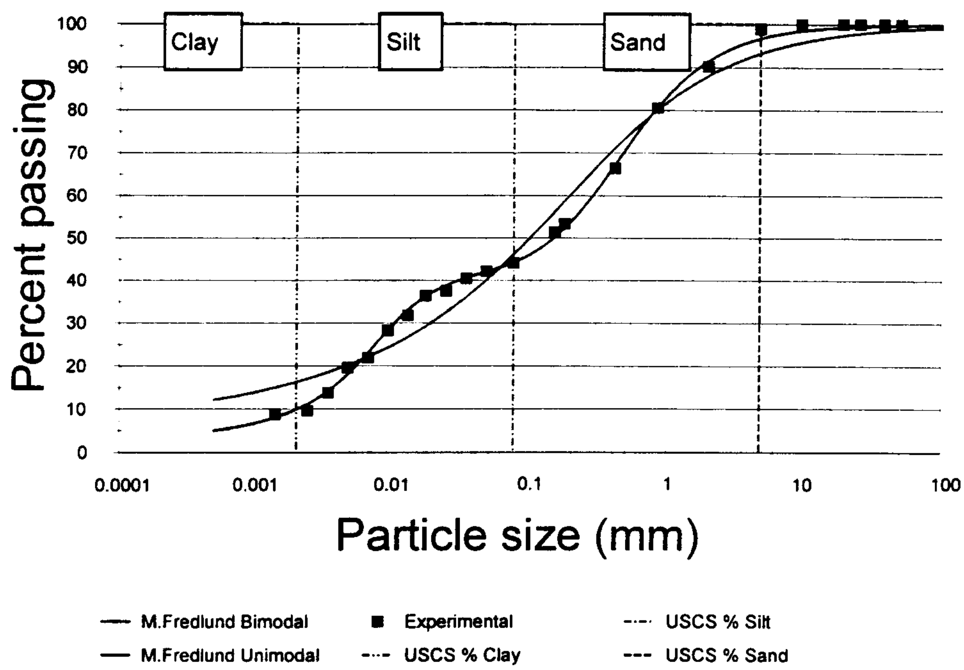

Fig. 9 Example of a bimodal fit on a gap-graded Saprolitic Soil tested at the University of Saskatchewan ($R^2 = 0.999$)

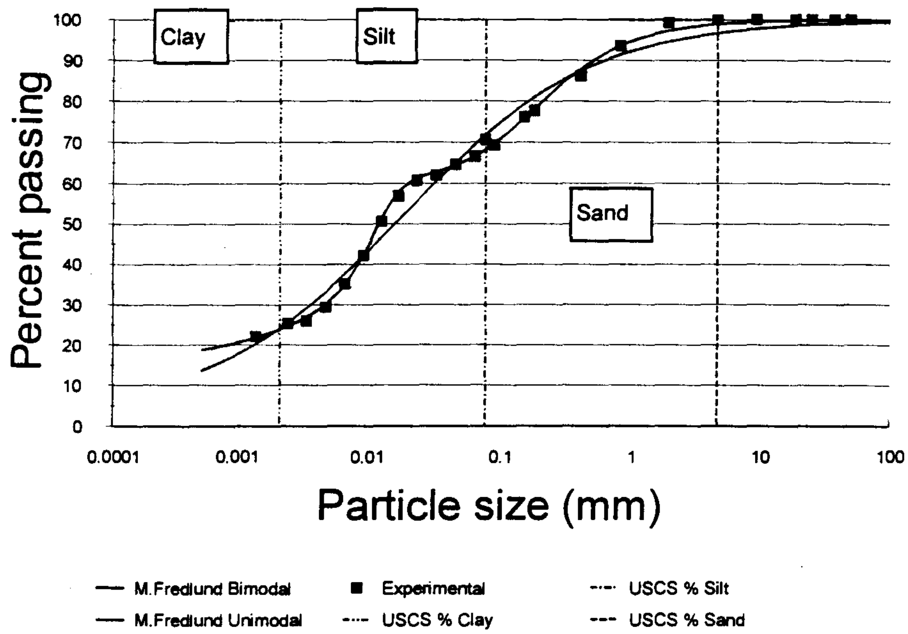

Fig. 10 Example of a bimodal fit of a gap-graded Saprolitic Soil tested at the University of Saskatchewan ($R^2 = 0.999$)

1.7 Application of the Mathematical Function for the Grain-size Distribution

The grain size distribution has been used extensively for the classification of soils. The application of the mathematical equations in this paper can be applied to geotechnical engineering practice. The use of equations to fit the grain-size distribution provides several advantages. Firstly, the equations presented in this paper provide a method for estimating a continuous function. Secondly, quantification of soils based on their grain size distribution is possible when equations are fit to datasets of soils information. Thirdly, equations provide a consistent method for determining physical indices such as percent clay, percent sand, percent silt, and particle diameter variables such as d_{10}, d_{20}, d_{30}, d_{50}, and d_{60}.

It has also been found that the grain size distribution is central to most methods of estimating the soil-water characteristic curve (Gupta and Larson, 1979; Arya and Paris, 1981; Haverkamp and Parlange, 1986, Ranjitkar and Sunder, 1989). An accurate representation of the soil particle sizes is essential when the grain-size distribution curve is used as the basis for the estimation of the soil-water characteristic curve. The equations presented in this paper appear to provide an excellent basis for the estimation of the soil-water characteristic curve (Fredlund et. al, 1997).

1.7.1 Parameters of the grain-size distribution equations

The unimodal fit of the grain-size distribution has been fit to many experimentally measured grain-size data sets extracted from research papers. The unimodal fit performed well with the exception of soils exhibiting bimodal behavior. The parameters of the unimodal equation vary in a manner similar to the parameters in the Fredlund and Xing (1994) soil-water characteristic curve equation. This study also investigated whether equation parameters could be grouped according to soil textural classifications. For example, is there a range of the n_{gr} parameter typical for silty sands? The results of this research indicate that *general* parameter groups can be identified but *specific* parameter groupings cannot be identified. The influence of equation parameters on each other does not allow for *specific* groupings. It was found that grouping soils is more successful when parameters with physical significance are selected. Successful groupings of soil properties has been achieved by combining soils according to physical parameters such as percent clay, percent silt, and percent sand and through the use of variables such as d_{10}, d_{20}, d_{30}, d_{50}, and d_{60}.

1.7.2 Determining physical parameters from the grain-size distribution equation

One of the benefits of the two grain-size equations presented in this paper is that conventional physical variables can be computed from the curves. The most commonly used variables are percent clay, percent sand, and percent silt. Also used are diameter variables such d_{10}, d_{20}, d_{30}, d_{50}, and d_{60}. The equations presented are of the form, $P_p(d)$ where d is particle diameter (mm). The percent clay, percent silt, and percent sand can therefore be a computed by substituting in the appropriate diameters. The diameters used depend upon the criteria associated with the various classification methods. The divisions can be determined for any classification method by substituting into the equations the appropriate diameters as shown in Fig. 11.

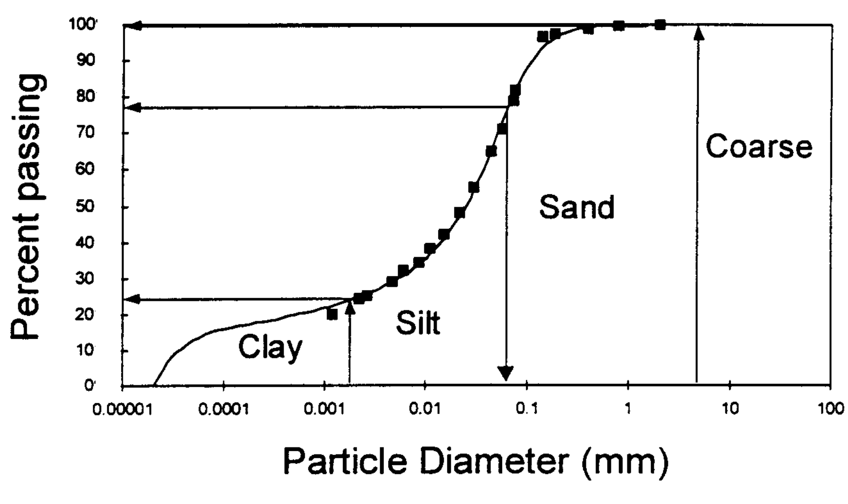

Fig. 11 Determination of the soil fractions (i.e., % clay, % silt, and % sand) when using the unimodal equation

The diameter variables must be computed in an inverse manner. The particle size diameter answers to the question,"What particle diameter has 10 percent of the total mass smaller than this size?" Taking the inverse of either the unimodal or bimodal equation is difficult. A half-length algorithm was therefore used to compute diameters from the grain size curve. An initial guess diameter was selected and the correction distance was progressively halved until the iteration process yielded a minimal error.

1.8 Conclusions

Unimodal and bimodal equations are presented to fit essentially any grain-size distribution dataset. The unimodal equation was found to provide a good fit of a variety of soils. The extremes of the grain-size distribution were also well-fit by the equation.

Gap-graded soils can be fit using a bimodal equation. The bimodal equation allows for a mathematical representation of any grain-size distribution where the sample contains two distinctly different, but dominant particle size groups.

Mathematical representation of the grain-size distribution provides numerous benefits. Curves can be identified and categorized. Likewise, the grain-size curves can be located in a data base using searching techniquies. Grain-size variables (i.e., *% clay*, d_{10}, d_{60}, etc.) can be mathematically determined from the equation. The unimodal and bimodal equations provide a method for fitting the three primary types of well-graded soils, uniform soils, and gap-graded soils.

The proposed continuous mathematical function for the grain-size curve sets the stage for further analysis to estimate the soil-water characteristic curve of a soil.

1.9 References

Arya, L. M., and Paris J. F., 1981, A physicoempirical model to predict the soil moisture characteristic from particle-size distribution and bulk density data, Soil Science Society of America Journal, Vol. 45, pp. 1023-1030.

Brooks R. H. and Corey A.T., 1964, Hydraulic Properties of Porous Media, Colorado State Univ. Hydrol. Paper, No. 3, 27, pp. March 1964.

Burdine, N. T., 1953, Relative permeability calculations from pore size distribution data, Journal of Petroleum Technology, Vol. 5, No. 3, pp. 71-78.

Campbell, G. S., 1985, Soil Physics with Basic, Elsevier, New York.

Fredlund, D. G., and Xing, A., 1994, Equations for the soil-water characteristic curve, Canadian Geotechnical Journal, Vol. 31, No. 3, pp. 521-532.

Fredlund, M. D., Fredlund, D. G., and Wilson, G. W., 1997, Prediction of the Soil-Water Characteristic Curve from Grain-Size Distribution and Volume-Mass

Properties, 3rd Brazilian Symposium on Unsaturated Soils, Rio de Janeiro, April 22-25.

Gardner, W. R., 1974, The permeability problem, Soil Science, 117, 243-249.

Gupta, S. C., and W. E. Larson, 1979, Estimating soil-water retention characteristics from particle size distribution, organic matter percent, and bulk density, Water Resources Research Journal, Vol. 15, No. 6, pp. 1633-1635.

Hagen, L. J., Skidmore, E. L. and Fryrear, D.W., 1987, Using two sieves to characterize dry soil aggregate size distribution, Transactions of the ASAE, 30(1), 162-165.

Haverkamp, R., and Parlange, J. Y., (1986), Predicting the water-retention curve from a particle-size distribution: 1. Sandy soils without organic matter, Soil Science, Vol. 142, No. 6, pp. 325-339.

Holtz, R. D. and Kovacs, W. D., 1981, An introduction to geotechnical engineering, Prentice-Hall, Inc., Englewood Cliffs, New Jersey.

Kemper, W. D. and Chepil, W. S., 1956, Size distribution of aggregates. In Methods of Soil Analysis, part 1. ed. C.A. Black Agronomy 9:499-510.

Kohnke, H., 1968, Soil Physics, McGraw-Hill Book Company, New York.

Mualem, Y., 1976, A new model for predicting the hydraulic conductivity of unsaturated porous media, Water Resources Res., 12, 513-522.

Ranjitkar S., and Sunder B., 1989, Prediction of hydraulic properties of unsaturated granular soils based on grain size data, Ph.D Thesis, University of Massachusetts, 75-131.

Shirizi M. A, and Boersma L., 1984, A unifying quantitative analysis of soil texture, Soil Science Society of America Journal, 48, 142-147.

van Genuchten, M. T., 1980, A closed form equation for predicting the hydraulic conductivity of unsaturated soils, Soil Science Society America Journal, pp. 892-890.

Wagner, L. E., and Ding, D., 1994, Representing aggregate size distributions as modified lognormal distributions, American Society of Agricultural Engineers, Vol. 37, No. 3, pp. 815-821.