General partial differential equation solvers for saturated-unsaturated seepage

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ABSTRACT: This paper presents the application of a general purpose partial differential equation solver, called PDEase2D and the associated soil property functions to the analysis of saturated-unsaturated seepage problem. Examples of steady state and transient seepage through an earth dam are used to illustrate the use of PDEase2D in solving saturated-unsaturated seepage problems.

1 INRODUCTION

Seepage analyses form an important and basic part of geotechnical engineering. The pore-pressure distribution in a soil are necessary to describe the flow processes and are also necessary to predict the volume change and shear strength changes associated with a change in stress state. Seepage through an unsaturated soil is mathematically characterized by a partial differential equation that is non-linear and soil properties that can be highly non-linear. As a result, the modeling of saturated-unsaturated soil systems has proven to be a challenge. There is need for numerical software packages that ensure convergence when solving seepage problems involving saturated-unsaturated soil systems (Fredlund, 1996).

This paper illustrates the use of a general-purpose partial differential equation solver, called PDEase2D, to solve seepage problems involving saturated-unsaturated soil systems. Several forms of permeability functions and water storage functions are used when solving the seepage problem. These forms are demonstrated through two example problems, one involving steady state seepage, and another involving transient seepage through unsaturated soil systems.

2 BACKGROUND

The governing partial differential equation for seepage through a heterogeneous, anisotropic, saturated-unsaturated soil can be derived by satisfying conservation of mass for a representative elemental volume, assuming that flow follows Darcy's law. If it is assumed that the total stress remains constant during a transient process and that pore-air pressure is atmospheric, the differential equation can be written as follows for the two-dimensional transient case:

$$\frac{\partial}{\partial x}\left(k_x \frac{\partial h}{\partial x}\right)+\frac{\partial}{\partial y}\left(k_y \frac{\partial h}{\partial y}\right)=m_2^w \gamma_w \frac{\partial h}{\partial t} \quad (1)$$

where h = total head (i.e., pore-water pressure head plus elevation head); k_x and k_y = coefficient of permeability of the soil in the x- and y-direction, respectively; γ_w = the unit weight of water (i.e., 9.81 kN/m^3); and $m^w{}_2$ = the slope of the soil-water characteristic curve (i.e., water storage).

For steady state seepage, only the coefficient of permeability is required because the time dependent term in eq. (1) disappears and the storage function drops out. The coefficient of permeability must be considered as a function of the stress state for an unsaturated soil. The coefficient of permeability can be written as follows for an unsaturated soil:

$$k_w = func[k_s, (\sigma-u_a), (u_a-u_w)] \quad (2)$$

where k_s = saturated coefficient of permeability; $(\sigma-u_a)$ = net normal stress; and (u_a-u_w) = matric suction. However, the coefficient of permeability of an unsaturated soil is predominantly a function of the matric suction.

Many permeability equations have been suggested in the literature and the following equations are used in this paper: Gardner (1958) equation; van Genuchten-Burdine (1980) equation; van Genuchten-Mualem (1980) equation; and Fredlund and Xing (1994) equation. In addition, linear interpolation between experimental data points is also used in this paper.

In addition to the non-linear coefficient of permeability function, the water storage function is also required for solving saturated-unsaturated transient seepage problems. The water storage indicates the amount of water taken or released by the soil as a result of a change in the pore-water pressure and it is the slope of the soil-water characteristic curve. Therefore, the water storage function is obtained by differentiating the soil-water characteristic curve with respect to the matric suction. Several equations have been proposed to describe the soil-water characteristic curve. These equations involve finding best-fit parameters, which produces a curve that fits the measured data. Any of the proposed equations can be used to describe soil-water characteristic curve, but only the van Genuchten (1980) equation and the Fredlund and Xing (1994) equation are used in this paper. The analysis of several soil types showed that an extended form of an error function equation can also approximate the water storage function. Therefore, an extended error function is also used in this paper. The solution of a transient seepage problem is obtained and compared with respect to van Genuchten (1980) equation, Fredlund and Xing (1994) equation, and extended error function.

3 PARTIAL DIFFERENTIAL EQUATION SOLVERS

In the last two decades, the development and application of the computer to solving complex problems has been extensive. Computer programs make use of numerical methods such as finite element technique. These computer programs have become necessary tools for solving engineering problems. An unsaturated soil problem involves the soil properties that are highly non-linear, such as coefficient of permeability and water storage functions. The partial differential equations to be solved become highly non-linear and require the input from persons specially trained in the area of mathematics. This has given rise to the use of general partial differential equation solvers that are designed to solve equations from many areas of engineering.

A finite element computer program, PDEase2D, marketed by Macsyma Corp, is one of the first general-purpose partial differential equation solvers to be marketed. Another similar software package called FlexPDE, is marketed by PDE Solution Ltd. While PDEase2D can analyze only two-dimensional problem, FlexPDE is extended to solve problems in three-dimensions. These software packages have several special features that are of interest to geotechnical engineers and have the potential to interface with many other software packages such as a computing program, MathCad, SoilVision (1998), graphics programs, AutoCAD and a database program, ACCESS.

The user of the general partial differential equation solver must specify the governing partial differential equation to be solved. The material properties can be described in the tabular form or as a mathematical equation. The boundary conditions can be specified as a dependent variable type (i.e., "head " type boundary condition) or a derivative of a dependent variable type (i.e., "flux" type boundary condition).

4 EXAMPLE PROBLEMS INVOLVING SEEPAGE THROUGH A SATURATED-UNSATURATED SOIL SYSTEM

A large number of example problems as well as a parametric study have been performed using the PDEase2D (Thieu, 1999). However, as part of this paper, only two example problems, the first associated with steady state seepage and the second associated with transient state seepage, will be presented. The soil material is assumed to be silt and isotropic with respect to the coefficient of permeability. Experimental data showing the coefficient of permeability versus matric suction, and the volumetric water content versus matric suction are obtained from Ho (1979). The saturated coefficient of permeability is 2.5 x 10^{-7} m/sec and the saturated volumetric water content is 0.381.

4.1 *Steady state seepage example*

This example is presented to illustrate the different forms of input the coefficient of permeability when analyzing steady-state seepage through an isotropic earth dam with a horizontal drain. The permeability functions used are the Gardner (1958) equation, van Genuchten-Mualem (1980) equation, van Genuchten-Burdine (1980) equation, Fredlund and Xing (1994) equation and a series of data points. The parametric study was done to find the best-fit values for the permeability functions using MathCad program. The permeability functions and their values of fitted parameters are shown in Table 1. Specified permeability functions together with the experimental data used to analyze the problem are presented graphically in Fig. 1.

The geometry, boundary conditions and finite element mesh used in running PDEase2D for steady-state seepage example are shown in Fig. 2. A maximum error of 0.1% was specified. The results of the total head distribution are presented in Fig. 3. The pore-water pressure distribution and flow vectors under steady state seepage are shown in Fig. 4 for different forms of data input. The position of the total head contours and phreatic lines are the same for all permeability functions.

The coefficient of permeability function does not need to be specified precisely when computing the distribution of pore-water pressure. Therefore, an approximate permeability function is adequate for

Table 1. Permeability functions and fitted parameters for steady-state seepage analyses

	Permeability functions	Fitted parameters
Gardner (1958)	$k_w = \frac{k_s}{1 + a\psi^n}$	$a = 1.969 \times 10^{-10}$, $n = 6.912$
van Genuchten-Burdine (1980)	$k_w = \frac{k_s}{\left[1 + (a\psi)^n\right]^{\left(1-\frac{2}{n}\right)}}$	$a = 4.127 \times 10^{-2}$, $n = 9.401$ $m = 0.787$
van Genuchten-Mualem (1980)	$k_w = \frac{k_s}{\left[1 + (a\psi)^n\right]^{\left(1-\frac{1}{n}\right)}}$	$a = 4.049 \times 10^{-2}$, $n = 8.852$ $m = 0.887$
Fredlund and Xing (1994)	$k_w = k_s \frac{\int_{ln(\psi)}^{b} \frac{\theta(e^y) - \theta(\psi)}{e^y} \theta'(e^y) dy}{\int_{ln(\psi_{aev})}^{b} \frac{\theta(e^y) - \theta_s}{e^y} \theta'(e^y) dy}$	$b = ln(1{,}000{,}000)$, y = dummy variable of integration representing the logarithm of suction, ψ_{aev} = soil suction at air entry value

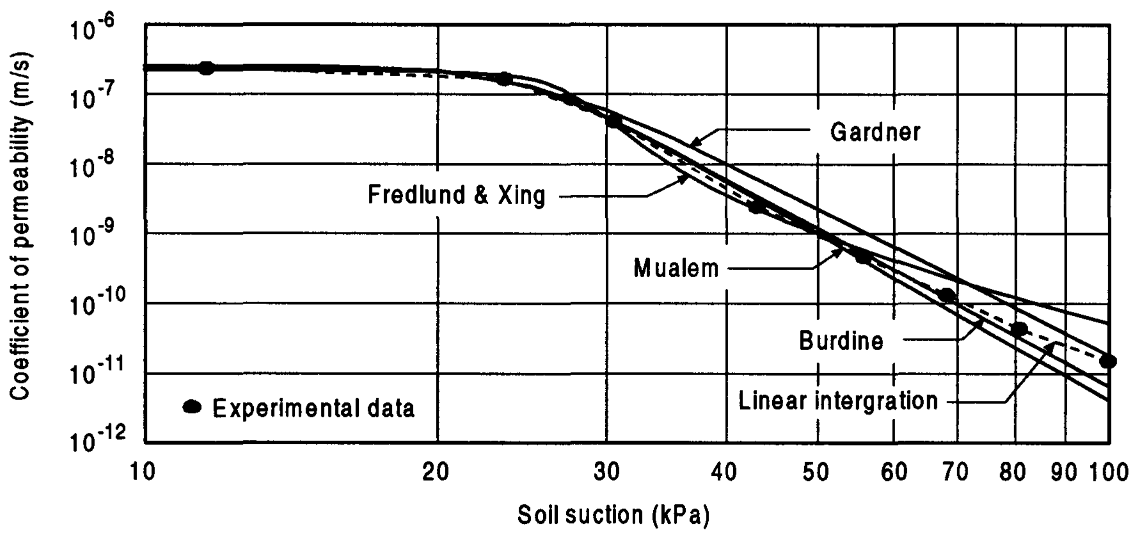

Figure 1. Specified permeability functions for the steady state seepage example

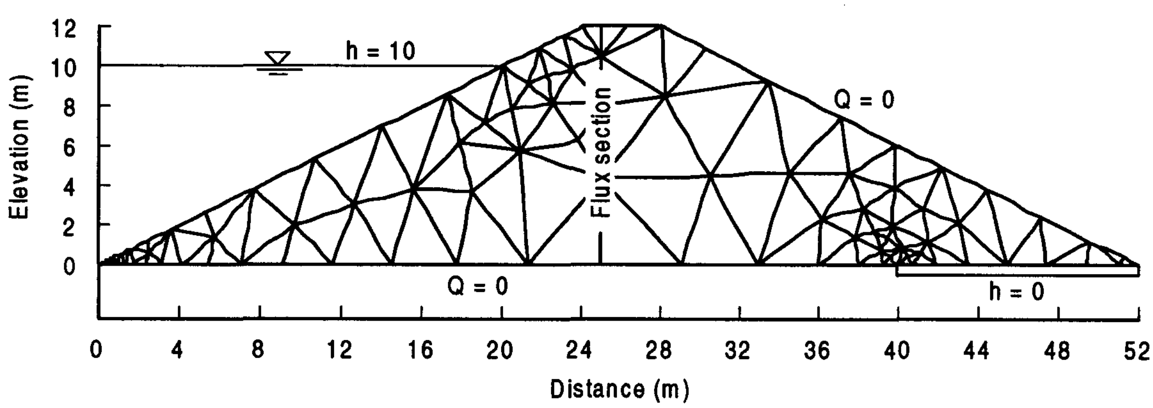

Figure 2. Geometry, boundary conditions and the finite element mesh for the steady state seepage example

analysis purposes. However, the quantity of flux through a section is slightly different, as shown in Table 2. The differences shown in Table 2 are not significant from an engineering standpoint and the results would indicate that any one of several possible permeability functions would yield satisfactory results.

4.2 *Transient state seepage example*

The second example is presented to show the different forms that can be used to input the coefficient of water storage when analyzing transient seepage through an isotropic earth dam with a horizontal drain. The base of the dam is selected as the datum.

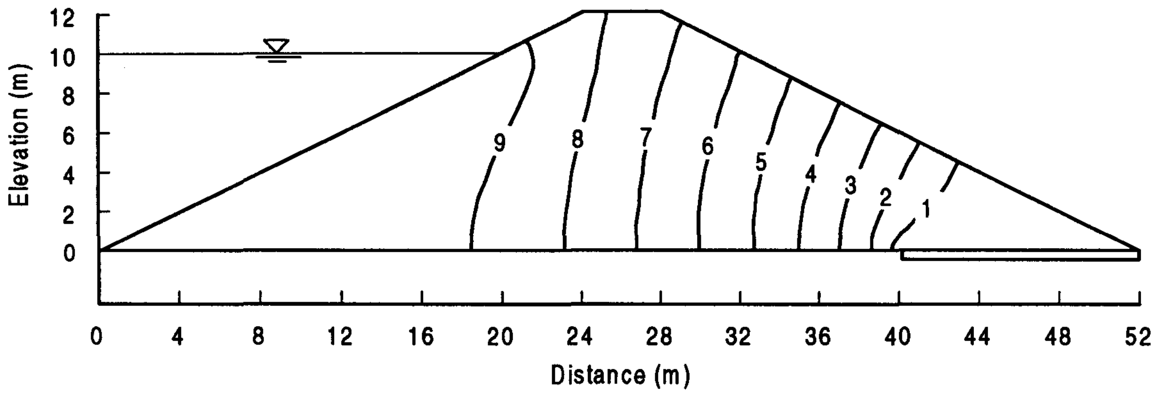

Figure 3. Total head distribution for steady state seepage example

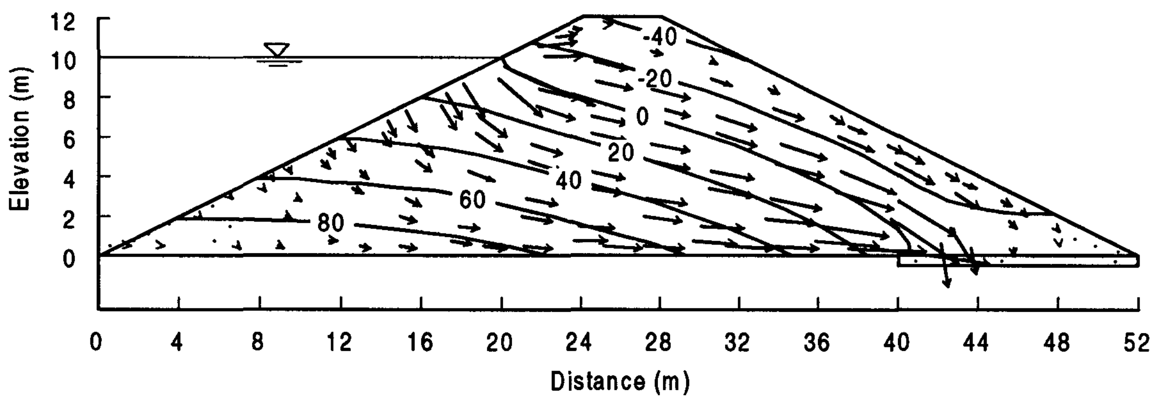

Figure 4. Pore-water pressure distribution and flow vectors for steady state seepage example

Table 2. Value of flux quantities using various permeability functions

Functions	Quantity of flux ($\times 10^{-7}$ m^3/s)	Deviation (%)
Gardner	7.449	0.8
van Genuchten-Burdine	7.432	0.5
van Genuchten-Mualem	7.430	0.5
Fredlund & Xing	7.376	0.2
Linear interpolation	7.272	1.6
Average	7.392	0

Initially, the dam is at steady-state conditions with the reservoir water level of 4 m above the datum. At a time assumed to be equal to zero, the water level in the reservoir is instantaneously raised to a level of 10 m above the datum.

Gardner (1958) equation is used to describe the permeability function. The water storage function is obtained by differentiating the soil-water characteristic curve. The soil-water characteristic curve is de scribed using the van Genuchten (1980) equation, the extended error function, the Fredlund and Xing (1994) equation and the step-wise values. The parametric study was done to find out the best-fit parameters for the water storage function using MathCad program. The software, SoilVision (1998) can also be used for this purpose. The derivatives of the soil-water characteristic curve with respect to matric suction and their best-fit values are shown in Table 3 and Fig. 5.

The geometry and boundary conditions for this example are shown in Fig. 6. A steady-state condition with water level of 4 m is first performed in order to get the distribution of initial pore-water pressure head. A maximum error limit of 0.5% was specified. The number of elements and nodes used in running this process varies with time from 181 to 1006, and from 428 to 2215, respectively. The finite element mesh at time equal to 15 hours is presented in Fig. 7.

The pore-water pressure distributions and the phreatic lines for elapsed time equal to 15 hours corresponding to different forms of inputting the data are shown in Fig. 8. The results indicate that the water storage functions defined are quite sensitive to the transient state seepage predictions at early times. The phreatic line obtained using various water storage functions would be closer at latter time steps, and approaches the same location at steady state conditions. The solution of the problem at steady state condition was presented previously in Figs. 4 and 5.

5 CONCLUSION

General partial differential equation solvers, such as PDEase2D, written with a focus on the mathematical aspects of the problem, show great potential for solving saturated-unsaturated seepage problems.

Table 3. Water storage functions and fitted parameters for transient seepage analyses

	Water storage functions	Fitted parameters
van Genuchten (1980)	$\frac{d\theta}{d\psi} = -\frac{\theta_s}{(1+(a\psi)^n)^m}(a\psi)^n \frac{mn}{\psi(1+(a\psi)^n)}$	$a = 4.391 \times 10^{-2}$, $n = 46.754$, $m = 0.036$
Extended error function	$\frac{d\theta}{d\psi} = Ce^{a(\lvert\psi-\omega\rvert)^n}$	$a = -0.183$, $n = 1.073$, $c = 0.028$, $\omega = 28.783$
Fredlund & Xing (1994)	$\frac{d\theta}{d\psi} = -\theta_s \left[\frac{1}{\ln\left[e+\left(\frac{\psi}{a}\right)^n\right]}\right]^m \frac{m}{\ln\left[e+\left(\frac{\psi}{a}\right)^n\right]}\left(\frac{\psi}{a}\right)^n \frac{n}{\psi\left[e+\left(\frac{\psi}{a}\right)^n\right]}$	$a = 26.127$, $n = 14.030$, $m = 0.622$

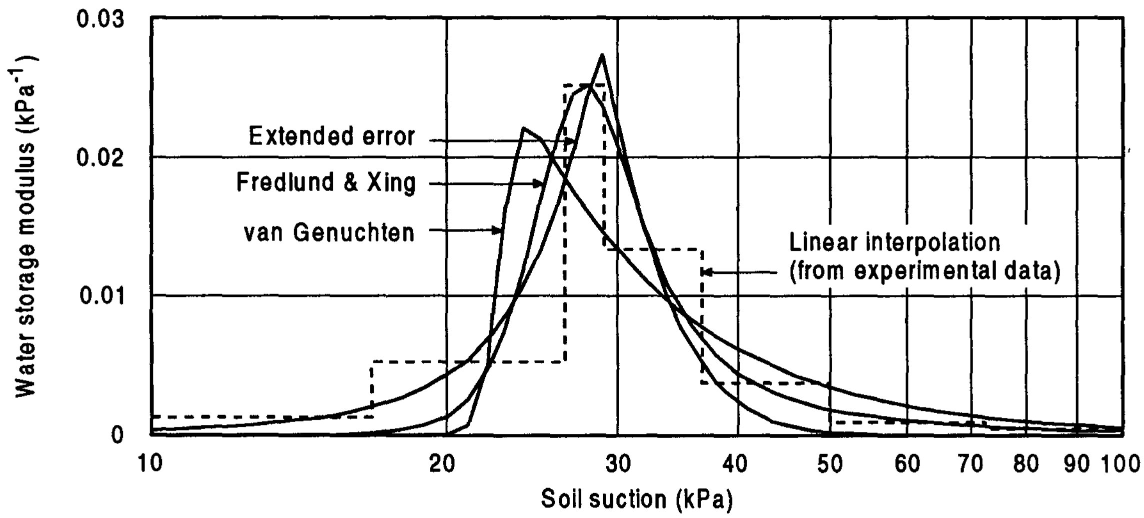

Figure 5. Specified water storage functions for transient seepage example

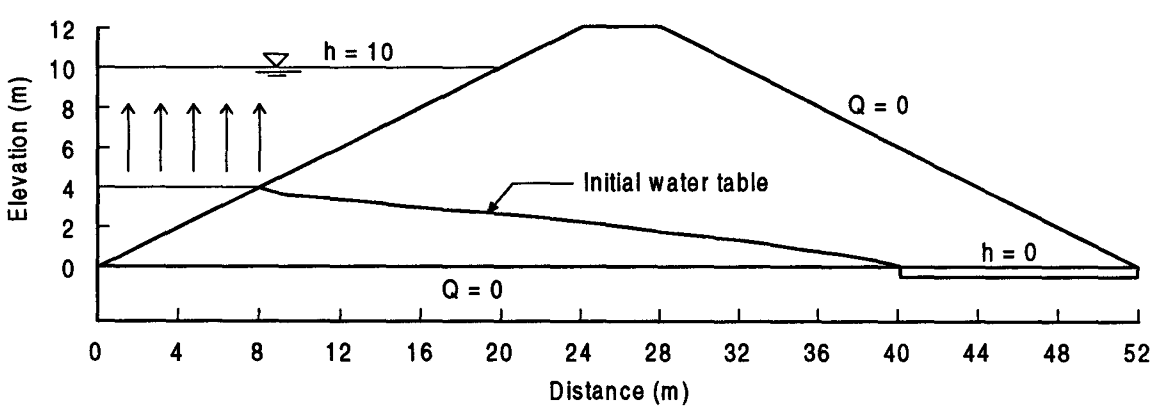

Figure 6. Geometry and boundary conditions for transient seepage example

These programs are particularly well-suited for solving unsaturated soils problems because of the attention given to: i) ensuring convergence when solving non-linear equations; ii) allowing material properties to be input in a variety of forms; and iii) allowing material properties to be non-linear in character.

It is possible to use a variety of formats for the input of soil property functions the PDEase2D. The formats for data input can vary from being a series of data points to a closed-form mathematical equation. In addition, MathCad and SoilVision software can be used in conjunction with PDEase2D to compute acceptable mathematical functions for unsaturated soil properties.

In general, it has been concluded, based upon the results of this study, that the philosophical approach behind this study is particularly valuable to the field

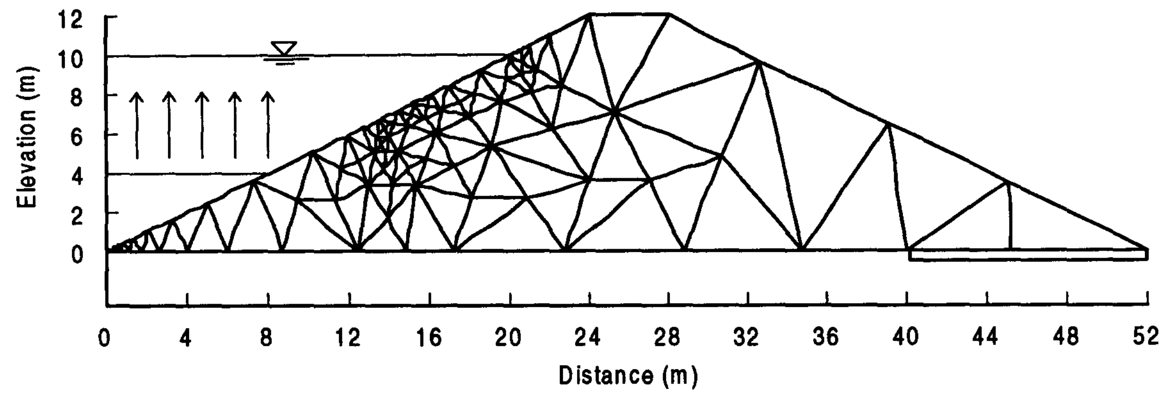

Figure. 7 Finite element mesh at an elapsed time, T = 15 hr, transient seepage example

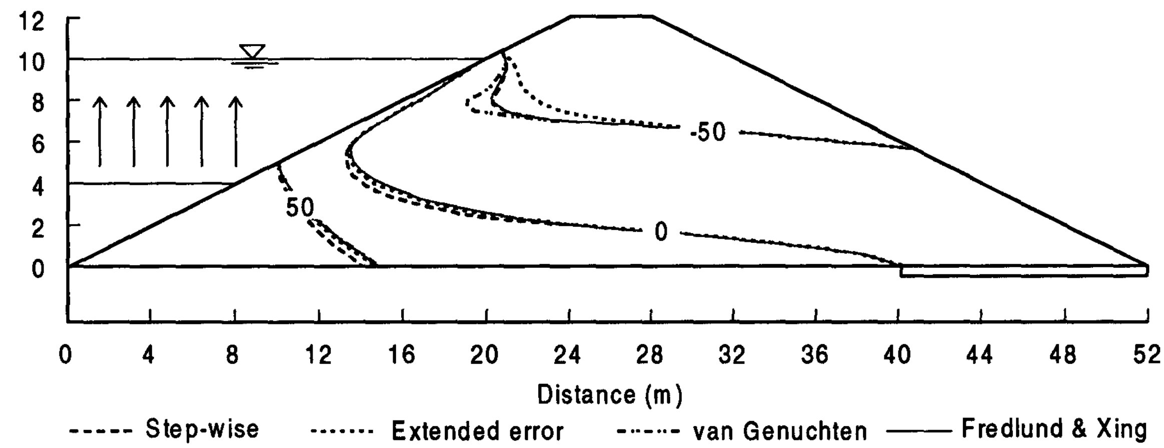

Figure 8. Comparison of the location of pore-water pressure contours for various water storage functions, after an elapsed time, T = 15 hr, transient seepage example

of unsaturated soil mechanics. The philosophical approach infers that optimum use should be made of science and math specialties that overlap with the needs in geotechnical engineering. Unsaturated soil analyses are complex and demanding. Therefore, the research work associated with related disciplines in mathematics and computing should be adopted wherever possible. This approach should prevent the ongoing "re-discovery" of numerical software solutions.

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