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The Use of a Rheological Model for the Visualization of Unsaturated Soils Processes

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ABSTRACT : *A rheological model is used to serve the role of aiding in visualizing the transient flow processes in an unsaturated soil subjected to various types of loading. The loadings may be due to an applied load or a change in the boundary flux or boundary pore pressure conditions. Two* ***Linear*** *rheological elements connected in series were used to simulate each layer of an unsaturated soil in a strata with* ***n*** *layers. Two systems of* ***n*** *linear differential equations were required to characterize the flow processes in the unsaturated soil. Solutions to the two systems of linear differential equations are determined from the eigenvalues of the systems.*

INTRODUCTION

The primary value of rheological models lies in the role they serve as an aid in visualizing soil behavior. For example, a rheological model can illustrate how a load applied to a saturated soil is initially carried by the water phase and is slowly transferred to the soil structure as the pore-water pressure dissipates.

THE PROPOSED RHEOLOGICAL MODEL FOR UNSATURATED SOILS

Two **Linear** rheological elements connected in series, one for the water phase and the other for the air phase were used to simulate the behavior of a layer of unsaturated soil (Fig. 1). The entire unsaturated soil strata can be simulated by interconnecting the rheological elements for both the air and water portions of the soil of subsequent layer, in series. In a non-homogeneous soil strata, the division of the strata into layers allows for the use of different soil parameters for each soil type.

Fredlund and Morgenstern (1977), showed on the basis of multiphase continuum mechanics principles that it was possible to use any two of the three possible stress state variables (i.e., $(\sigma - u_w)$, $(\sigma - u_a)$ and $(u_a - u_w)$) in order to describe soil behavior. When developing a rheological model to simulate unsaturated soil behavior, it appears that the $(\sigma - u_w)$ and $(\sigma - u_a)$ combination has advantages over other combinations.

The Hookean constants, c_w and c_a, account for the compressibility of water and air, respectively. The fluids in the dashpots are assumed to be incompressible. The Hookean spring constants, κ_w and κ_a, account for the compressibility of soil structure with respect to each of the stress state variables, $(\sigma - u_w)$ and $(\sigma - u_a)$, respectively, where σ is the total stress applied, u_w is the pore-water pressure and u_a is the pore-air pressure.

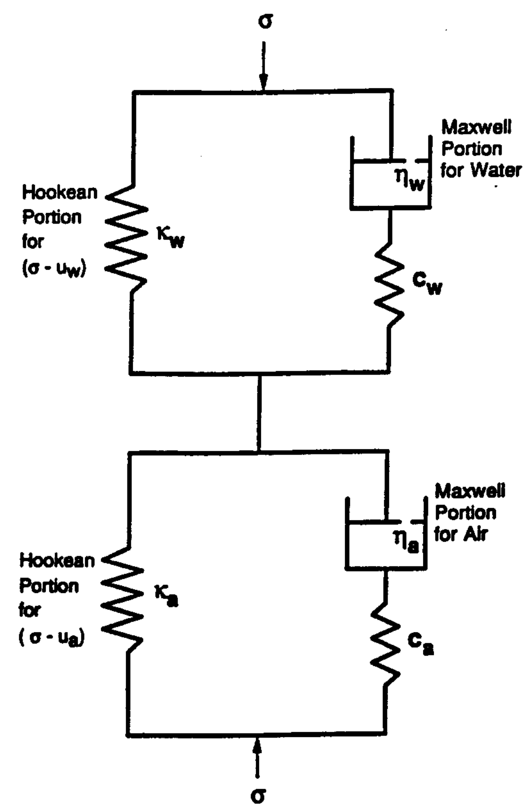

Fig. 1 Two **Linear** rheological elements in series representing the $(\sigma - u_w)$ and $(\sigma - u_a)$ stress state variables.

One **Linear** rheological element is required to model the stress state variable which involves the solid and the water phase (i.e., $(\sigma - u_w)$) and another to model the stress state variable which involves the solid and the air phase (i.e., $(\sigma - u_a)$). The two **Linear** elements are connected in series as the two stress state variables are independent.

The hydraulic conductivity properties of the soil with respect to water and air are represented by the viscosity constants η_w and η_a, respectively. The total strain of a unit infinitesimal layer is denoted by ε. The strain of the **Linear** rheological elements for the water phase and the air phase are denoted by ε_w and ε_a, respectively.

BOUNDARY CONDITIONS

The total stress applied to a **Linear** rheological model acts equally on both of the elements associated with each of the two stress state variables, $(\sigma - u_w)$ and $(\sigma - u_a)$, respectively.

The stress in the Maxwell and Hookean portions of each of the **Linear** elements must always add up to the total stress applied to satisfies the conservation of energy requirement (i.e., $\sigma = \sigma_H + \sigma_M$, where σ_H = stress on the Maxwell portion of the **Linear** element, σ_M = stress on the Hookean portion of the **Linear** element which could represent either the pore-water pressure, u_w, or the pore-air pressure, u_a, σ = total stress applied to the **Linear** element).

The total strain, ε, is always equal to the sum of the strain in the **Linear** element associated with the $(\sigma - u_w)$ stress state variable, ε_w, and the strain in the **Linear** element associated with the $(\sigma - u_a)$ stress state variable, ε_a (i.e., $\varepsilon = \varepsilon_w + \varepsilon_a$).

The Hookean portion and the Maxwell portion of each **Linear** rheological element must always deform equal amounts

CONSTITUTIVE RELATIONS

First, consider only the **Linear** rheological element associated with the stress state variable, $(\sigma - u_w)$. The total stress applied can be written in terms of the stress in the Hookean portion plus the stress in the Maxwell portion of the model:

$$\sigma = \sigma_H + \eta_w(\dot{\varepsilon}_w - \dot{\varepsilon}_c) \qquad (1)$$

where ε_c= strain of the Hookean spring of the Maxwell portion of the model which is related to the compressibility of water, ε_w= strain in the dashpot due to the flow of water, $\dot{\varepsilon}_w$ and $\dot{\varepsilon}_c$ = the derivatives of ε_w and ε_c, respectively, η_w= dashpot constant for the water phase.

Since $\sigma_H = \kappa_w \varepsilon_w$ and $\sigma_M = c_w \varepsilon_c$ (where κ_w = Hookean spring constant related to the compressibility of the soil with respect to a change in $(\sigma - u_w)$ and c_w = Hookean spring constant related to the bulk compressibility of water), Eq. (1) can be re-arranged to give a rate of strain condition for the **Linear** rheological element associated with $(\sigma - u_w)$:

$$\dot{\varepsilon} = \frac{\dot{\sigma}_M}{c_w} = \frac{\dot{\sigma} - \dot{\sigma}_H}{c_w} = \frac{0 - \dot{\sigma}_H}{c_w} = -\frac{\kappa_w}{c_w}\dot{\varepsilon}_w \qquad (2)$$

A similar rate of strain condition can be written for the **Linear** rheological element associated with $(\sigma - u_a)$.

DIFFERENTIAL EQUATIONS FOR THE LINEAR RHEOLOGICAL ELEMENTS OF A SINGLE LAYER

The stress-strain equation for the **Linear** rheological element associated with $(\sigma - u_w)$ can be written as:

$$\varepsilon_w = \frac{\sigma}{\kappa_w + c_w}\left\{1 + \frac{c_w}{\kappa_w}\left[1 - e^{-\frac{\kappa_w c_w}{\eta_w(\kappa_w + c_w)}t}\right]\right\} \qquad (3)$$

The stress-strain equation for the **Linear** rheological element associated with $(\sigma - u_a)$ can be written as:

$$\varepsilon_a = \frac{\sigma}{\kappa_a + c_a}\left\{1 + \frac{c_a}{\kappa_a}\left[1 - e^{-\frac{\kappa_a c_a}{\eta_a(\kappa_a + c_a)}t}\right]\right\} \qquad (4)$$

MATHEMATICAL MODEL FOR A MULTI-LAYER SYSTEM

A transient flow process such as consolidation takes place as a result of the application of a total load to a soil. In an unsaturated soil with one way drainage through the top, both water and air would move upward from a lower layer to an upper layer. The water phase and the air phase are assumed to be continuous throughout the soil. For illustration, this process is simulated using the 3-layer system shown in Fig. 2.

The **Linear** rheological elements associated with the stress state variables, $(\sigma - u_w)$ and $(\sigma - u_a)$, in the n-layer strata can be separated into two parts. An equivalency for the **Linear** rheological elements is shown in Fig. 3. For simplicity, the compressibility of the air phase and the viscosity of the water phase are assumed linear in this paper.

The compressibility of water and air is simulated using Hookean springs. In both the water and air phases, only the bottom layer is truly a **Linear** rheological element. The other layers are a slight modification of the **Linear** rheological element in that the pressure in one piston is fed into the adjacent piston in a series manner. Based on Eq. [2], the water phase differential equations can be written as:

$$\dot{\varepsilon}_i = -\frac{\kappa_w}{c_w}(\dot{z}_i - \dot{z}_{i+1}),\ i = 1, 2, \ldots, n-1 \qquad (5)$$

$$\dot{\varepsilon}_n = -\frac{\kappa_w}{c_w}(\dot{z}_n - 0) \qquad (6)$$

where $z_1(t)$, $z_2(t)$, ..., $z_n(t)$ denote the positions of the **n** pistons, respectively (see Fig. 3) and $\varepsilon_1(t)$, $\varepsilon_2(t)$, ..., $\varepsilon_n(t)$ denote the strains of the **n** pistons, respectively.

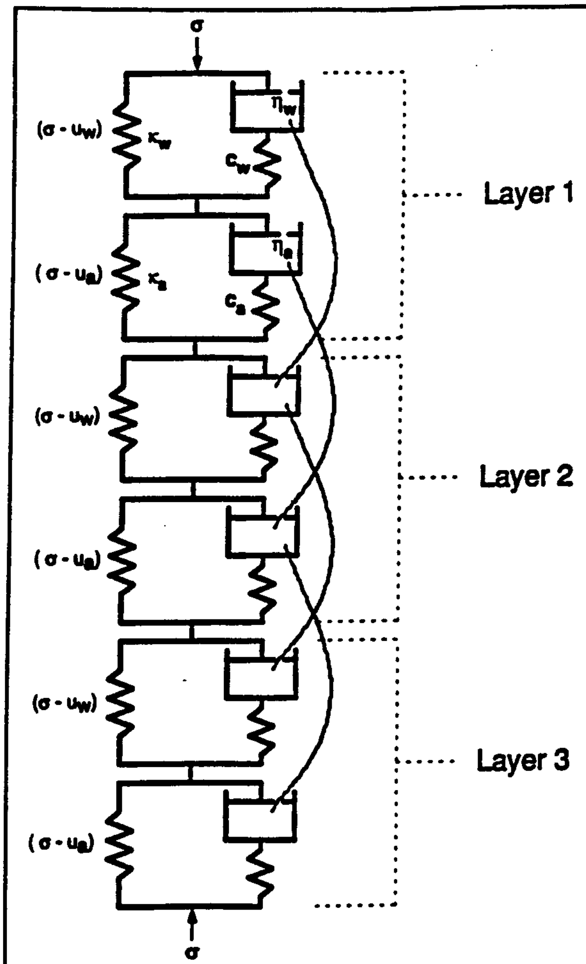

Fig. 2 A 3-layer system demonstrating 1-D consolidation process in an unsaturated soil.

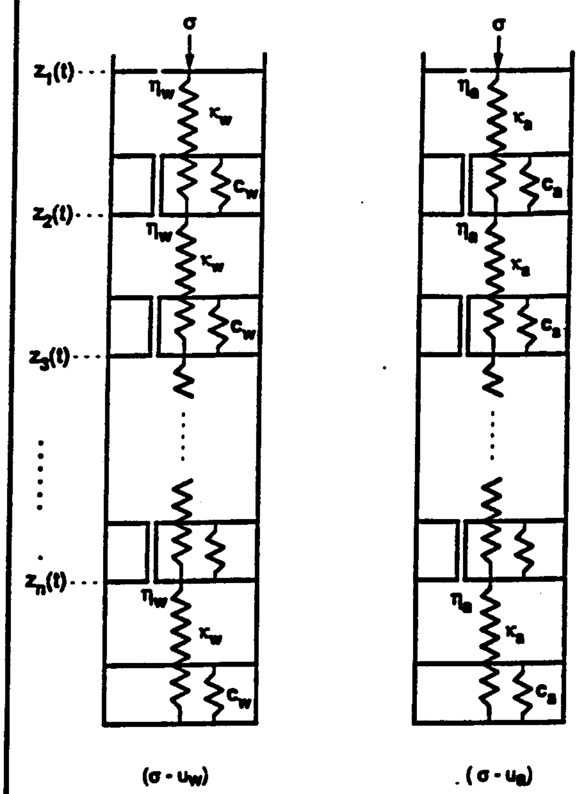

Fig. 3 Proposed rheological models for the $(\sigma - u_w)$ and $(\sigma - u_a)$ stress state variables.

In the first layer of the **Linear** rheological element associated with the stress state variable, $(\sigma - u_w)$, the total stress, σ, is divided between the pore-water pressure and the stress acting in the spring representing the soil structure. That is,

$$\sigma = \kappa_w(z_1 - z_2) + \eta_w(\dot{z}_1 - \dot{\varepsilon}_1 - \dot{\varepsilon}_2 - \ldots - \dot{\varepsilon}_n) \tag{7}$$

Substituting Eqs. [5] and [6] into the conservation equation (i.e., Eq. [7]) gives:

$$\sigma = \kappa_w(z_1 - z_2) + \eta_w \frac{c_w + \kappa_w}{c_w} \dot{z}_1 \tag{8}$$

The total stress, σ, also acts on the second piston associated with the **Linear** rheological element for the stress state variable, $(\sigma - u_w)$, of the second layer. However, the Hookean stress, $\kappa_w(z_1 - z_2)$, developed in the first layer is the only stress which causes the piston in the element of the second layer to move downward. This stress is again divided between the spring stress and pore-water pressure increment in the second layer. The rates of strain for the second layer can be rewritten in terms of the water phase dashpot constant and the spring constants for the water and the soil structure as:

$$\kappa_w(z_1 - z_2) = \kappa_w(z_2 - z_3) + \eta_w \frac{c_w + \kappa_w}{c_w} \dot{z}_2 \tag{9}$$

Similar equations can be written for the third and subsequent layers. For the *n*th layer:

$$\kappa_w(z_{n-1}-z_n)=\kappa_w z_n+\eta_w\frac{c_w+\kappa_w}{c_w}\dot{z}_n \tag{10}$$

Equations [8] to [10] represent a system of linear differential equations for the Linear rheological element associated with the stress state variable, (σ - u_w), of the unsaturated soil. The system of linear differential equations for the response of the **Linear** rheological element associated with the stress state variable, (σ - u_w), can be written in matrix form as:

$$\dot{z}=Az+b,\ z(0)=z^0 \tag{11}$$

where

$$A=\frac{\kappa_w c_w}{\eta_w(\kappa_w+c_w)}\begin{bmatrix}-1 & 1 & 0 & 0 & \cdots & 0\\ 1 & -2 & 1 & 0 & \cdots & 0\\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\ 0 & \cdots & \cdots & 1 & -2 & 1\\ 0 & \cdots & \cdots & 0 & 1 & -2\end{bmatrix},\ b=\frac{\sigma c_w}{\eta_w(\kappa_w+c_w)}\begin{bmatrix}1\\0\\ \cdot\\ \cdot\\ 0\end{bmatrix},\ z^0=\frac{\sigma}{\kappa_w+c_w}\begin{bmatrix}n\\ n-1\\ \cdot\\ \cdot\\ 1\end{bmatrix}$$

The non-zero initial condition (i.e., $z(0)=z^0$) is due to the compressibility of water. If water is assumed to be incompressible, the c_w term goes to infinity and z_0 becomes a zero vector and the system of linear differential equations (i.e., Equation [11]) reduces to the equation for the Kelvin model for saturated soils (Xing, Fredlund and Gan, 1995).

The system of linear differential equations for the **Linear** rheological element associated with the stress state variable, (σ - u_a), has the same form as Eq. [11] for the stress state variable, (σ - u_w), with all the subscript *w* replaced by the subscript *a*..

The total strain for the unsaturated soil is the summation of the solutions for the **Linear** rheological elements associated with the stress state variables, (σ - u_w), and (σ - u_a). The total strain, $\varepsilon(t)$, is given by,

$$\varepsilon(t)=\frac{z_{w1}(t)+z_{a1}(t)}{z_{w1}(\infty)+z_{a1}(\infty)} \tag{12}$$

where $z_w(t)=(z_{w1}(t),\ ...,\ z_{wn}(t))$ and $z_a(t)=(z_{a1}(t),\ ...,\ z_{an}(t))$ denote the solutions for the (σ - u_w) and (σ - u_a) portions, respectively; $z_{wl}(\infty)$ and $z_{al}(\infty)$ denote the limits of $z_{wl}(t)$ and $z_{al}(t)$, respectively, as time goes to infinity.

The pore-water pressure, $u_w(t)$, and the pore-air pressure, $u_a(t)$, in the *i*-th layer are respectively given by the following Eqs. [13] and [14].

$$u_{wi}(t)=\sigma-\kappa_w\left[z_{wi}(t)-z_{w(i+1)}(t)\right] \tag{13}$$

$$u_{ai}(t)=\sigma-\kappa_a\left[z_{ai}(t)-z_{a(i+1)}(t)\right] \tag{14}$$

CLOSED-FORM SOLUTION FOR THE RHEOLOGICAL MODEL

The closed-form solution for Eq. [11] can be expressed as.

$$z(t)=\begin{bmatrix}z_1(t)\\ z_2(t)\\ \cdot\\ \cdot\\ z_n(t)\end{bmatrix}=e^{At}z^0+\int_0^t e^{A(t-s)}b\,ds \tag{15}$$

The coefficient matrix, A, of the initial value problem given by Eq. [11] is tri-diagonal and satisfies the condition of a Jordan matrix. According to the Sturm theorem (Dickson, 1939), A has n real distinct eigenvalues. The n corresponding eigenvectors form the fundamental matrix for the system of differential equations (i.e., Eq. [11]). Using matrix theory, it can be further proved that all the eigenvalues are negative. The solution of Eq. [11] has the following form:

$$z_1(t) = \alpha_0 + \sum_{i=1}^{n} \alpha_i e^{\lambda_i t} \quad (16)$$

where:

α_i = constant (i = 0, 1, ..., n)
λ_i = eigenvalue of A ($\lambda_i < 0$; i = 1, 2, ..., n)

Equation [15] implies that the consolidation process is primarily governed by the eigenvalues of A. The n eigenvalues of A are uniquely determined by the soil parameters since only κ_w and η_w appear in the coefficient matrix, A. The external load or stress, σ, only affects the magnitude of the strain.

NUMERICAL SOLUTION FOR THE PROPOSED RHEOLOGICAL MODEL

A numerical solution was formulated for solving the two systems of linear differential equations using the Runge-Kutta method. The unsaturated soil simulated using the rheological model does not have to be homogeneous. Each layer can have its own soil parameters. A numerical solution is preferred if the soil is divided into a large number of layers.

COMPARISON OF THE RHEOLOGICAL MODEL WITH EXPERIMENTAL DATA

The rheological model was compared to experimental results for a compacted kaolin. The compacted kaolin has an initial dry unit weight of 13.185 kN/m^3, an initial void ratio of 1.0696, an initial water content of 34.32% and an initial degree of saturation of 78.9% (data from Fredlund and Rahardjo, 1993). The soil specimen was simulated by a 10-layer rheological model using approximate soil parameters obtained from a consolidation test where the total stress was increased from 101.1 kPa to 202.2 kPa.

The water phase constitutive relation can be written as:

$$\frac{dV_w / V_o}{d(\sigma - u_a)} = m_1^w + m_3^w \frac{d(\sigma - u_w)}{d(\sigma - u_a)} \quad (17)$$

where dV_w/V_o = change in the volume of water in the soil specimen with respect to the initial volume of the soil specimen, m_1^w = coefficient of water volume change with respect to a change in net normal stress, m_3^w = coefficient of water volume change with respect to a change in the stress state variable, $(\sigma - u_w)$.

The following relationships for the water phase rheological model can be inferred:

$$\eta_w = \frac{\gamma_w}{k_w} \Delta z, \quad k_w \frac{1}{\Delta z} \left(\frac{1}{m_1^w + m_3^w} \right) \quad (18)$$

where k_w = coefficient of permeability with respect to water, γ_w = unit weight of water, Δz = the thickness of each of the layers.

Similarly, for the air phase, the relationships are:

$$\eta_a = \frac{\gamma_a}{k_a} \Delta z, \quad k_a \frac{1}{\Delta z} \left(\frac{1}{m_1^a + m_3^a} \right) \quad (19)$$

where k_a = coefficient of permeability with respect to air, γ_a = unit weight of air, m_1^a = coefficient of air volume change with respect to a change in net normal stress, m_3^a = coefficient of air volume change with respect to a change in the stress state variable, $(\sigma - u_a)$.

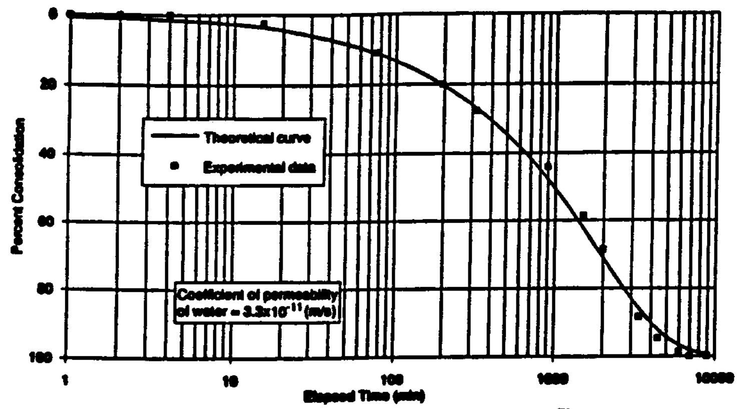

Fig. 4 Percent consolidation versus logarithm of time plot for compacted kaolin (Fredlund and Rahardjo, 1993).

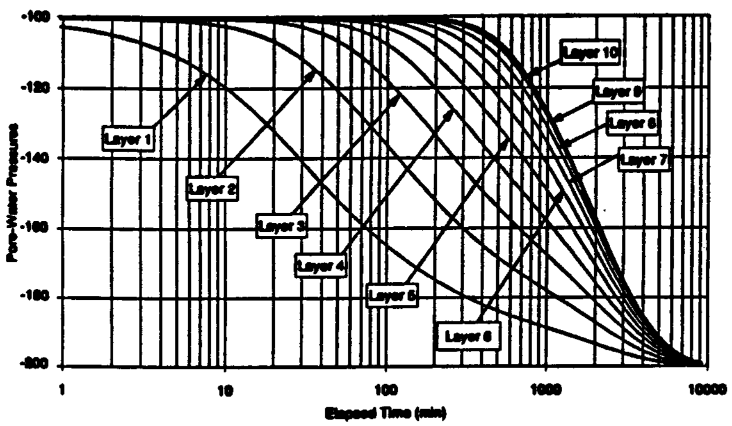

Fig. 5 Pore-water pressures in the compacted kaolin, from Linear rheological model.

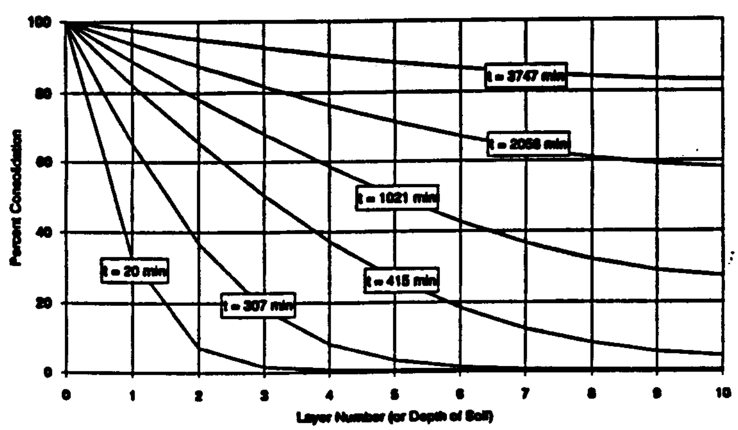

Fig. 6 Percent consolidation at different layers in the compacted kaolin, from Linear rheological model.

The percent consolidation versus the logarithm of time plot is shown in Fig. 4. The pore-water pressures in each layer as a function of the logarithm of time are shown in Fig. 5. The percent consolidation versus the depth into the soil is shown in Fig. 6. It can be seen, from Fig. 4, that the theoretical curve computed using the rheological model (i.e., Eq. [11]) is in good agreement with the experimental data.

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