prefailure yielding. The five-level nested successive iteration procedure may require several tens of thousands of iteration steps for each factor of safety calculation. It may also be prone to divergence problems in some cases. These aspects of the rigorous 3D method may delay its use as a practical tool.

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A general limit equilibrium model for three-dimensional slope stability analysis: Reply[1]

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The subject of three-dimensional numerical analysis (i.e., either finite element or limit equilibrium) is becoming of increased importance to geotechnical engineering as a result of rapid developments in computing capabilities. Hungr has done research on three-dimensional slope stability theories and related software development for several years and his points of discussion are certainly welcome. The authors agree that more research needs to be done on the subject of three-dimensional analysis. Part of the purpose of the above paper was to establish a theoretical basis for future research studies.

The authors' paper proposed a general limit equilibrium formulation for three-dimensional slope stability analysis. A method of solution was proposed for three-dimensional situations where the geometry could be simplified or where the intercolumn forces were known. The approach was an extension of the two-dimensional general limit equilibrium method (GLE) proposed by Fredlund and Krahn (1977) that was consistent with the approach proposed by Morgenstern and Price (1965). Both the GLE method and the Morgenstern–Price method are classified as rigorous methods in that both force and moment equilibrium are satisfied while allowing the flexibility of assuming any form for the interslice force function (Fredlund 1984).

The proposed three-dimensional, general limit equilibrium method satisfies both force and moment equilibrium and proposes the use of five intercolumn force functions (i.e., eqs. 1–5). Equations 1–4 indicate the relationship between the shear and normal forces on the sides of a column. Equation 5 indicates the relationship between the shear and normal force on the base of the column, in a direction orthogonal to the direction of movement. Vertical force equilibrium is satisfied for each column and provides a normal force equation (i.e., eq. 6). The factors of safety with respect to moment and force equilibrium are formulated in terms of the normal force N (i.e., eqs. 7 and 8).

The problem in solving the formulation arises as a result of the necessity to solve for five λ values. The questions raised by the discusser relate to the following: "How should the analyst solve for the five λ values and obtain the same factor of safety for moment and force equilibrium?" The authors chose to select several simplified three-dimensional geometries and show through the use of a three-dimensional stress analysis that several of the intercolumn force functions tend to zero for the selected cases. As a result, the three-dimensional analysis can be simplified provided the geometric simplifications are reasonably well satisfied. This is not to imply, however, that these intercolumn simplifications should be made for all three-dimensional situations.

Hungr has also indicated some assumptions that could be made to solve for all five λ values. He suggests that the intercolumn force functions of the sliding body should be determined from a three-dimensional finite element stress analysis. He then suggests that the factor of safety and the five λ values be solved with a five-level nested successive iterative procedure using eqs. [6]–[8] in the paper. He points out that the iterative procedure may require far too many iterations and as a result may dismiss the feasibility of a rigorous three-dimensional slope stability analysis in engineering practice. The authors agree with this conclusion for the case of five independent λ values. However, the authors suggest that while it may not be feasible to solve for five independent λ values, it is also not necessary to

[1]Discussion by O. Hungr, Canadian Geotechnical Journal, **31**: 793–795.

assume that the λ values are independent when the intercolumn force functions are defined from a three-dimensional stress analysis.

The λ values can be visualized as the percentage of intercolumn shear forces computed from a stress analysis that is required to solve for an equal moment and force equilibrium factor of safety. Previous studies (Fredlund 1984; Fan et al. 1986) have shown that for a simple two-dimensional slope consisting of a frictional material, the λ value approaches 1.0. In other words, 100% of the interslice shear forces computed from a stress analysis yield essentially the same factor of safety for both moment and force equilibrium. As the soil has an increasing cohesive component to strength, the λ value satisfying moment and force equilibrium decreases.

The above results from two-dimensional slope stability studies can be extended to the three-dimensional situation. If λ is used as a percentage of the intercolumn shear forces, as obtained from a three-dimensional stress analysis, then it can be reasoned that all λ values should be the same. This is particularly true for the λ values associated with the sides of the columns. The above rationale suggests that there is no theoretical justification for using varying percentages of the results of a stress analysis on different planes of the columns. To do so would contravene the principle of the conservation of energy.

Our increased computing capabilities are making both three-dimensional limit equilibrium analysis and three-dimensional stress analysis a possibility in engineering practice. It appears to be only a matter of a short time until the results of a stress analysis are imported into a limit equilibrium analysis. With this procedure, the stress state at any point within the sliding mass is defined at the base of each column. The shear stresses and the normal stress can be calculated directly from the stress state. At this point in the analysis, it is possible to take one of two approaches. One procedure is to use one λ value and retain the essence of the limit equilibrium approach. Alternatively, the activating force can be calculated from the shear stress, and the resisting force can be calculated from the normal stress together with the shear strength parameters. A comparison of the resisting and activating forces along the entire slip surface allows the estimation of the overall factor of safety of the sliding mass. As a result, it will no longer be necessary to define the intercolumn force functions and solve for the five λ values. Rather, the overall factor of safety becomes more directly based on the finite element stress analysis.

In the event that the results of a three-dimensional stress analysis are not available for the three-dimensional limit equilibrium analysis, it is necessary to assume appropriate intercolumn force functions for each surface of the column. In this case, it would appear to be most reasonable to scale the intercolumn force functions in an appropriate manner (i.e., based on the engineer's estimate of stress conditions) and then solve for one λ value. This is essentially what the authors were attempting to illustrate through the use of simple three-dimensional geometries. The examples showed the intercolumn force functions, which tend towards zero for these simple geometry situations. In other words, some of the intercolumn force functions can be given small values (or zero values) without significantly influencing the overall three-dimensional factor of safety.

Most of the points raised by Hungr in his discussion attempt to improve our understanding of the use of independent λ values. The authors agree with most of his comments but would prefer to view his comments as insight on the intercolumn force functions rather than insight on independent λ values. The intercolumn force function can be taken as zero when λ is zero and as a result either λ or the function can be used for visualization purposes.

Further study should be given as to how best to handle the shear force orthogonal to the primary direction of movement on the base of the column. When the direction of movement along the base of the column deviates from the direction used for analysis purposes, the analysis must also solve for a value for λ_5 or have some other method to solve for the shear on the base of the column (i.e., the authors agree with point 2 of the discussion). Stress conditions become even more complex when the movement of the soil mass changes direction throughout the mass. This complexity was not addressed in the paper by the authors.

In conclusion, the authors suggest that greater attempts should be made to pursue techniques for combining the results of a stress analysis with a limit equilibrium analysis. Such techniques are now feasible without being computationally excessive. At the same time, less effort should be made to second-guess what the stresses might be in a soil mass.

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