# Equations for the soil-water characteristic curve

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The soil-water characteristic curve can be used to estimate various parameters used to describe unsaturated soil behaviour. A general equation for the soil-water characteristic curve is proposed. A nonlinear, least-squares computer program is used to determine the best-fit parameters for experimental data presented in the literature. The equation is based on the assumption that the shape of the soil-water characteristic curve is dependent upon the pore-size distribution of the soil (i.e., the desaturation is a function of the pore-size distribution). The equation has the form of an integrated frequency distribution curve. The equation provides a good fit for sand, silt, and clay soils over the entire suction range from 0 to $10^6$ kPa.

*Key words*: soil-water characteristic curve, pore-size distribution, nonlinear curve fitting, soil suction, water content.

La courbe caractéristique sol-eau peut être utilisée pour estimer divers paramètres décrivant le comportement d'un sol non saturé. On propose ici une équation pour cette courbe caractéristique sol-eau. Un programme non linéaire, par moindres carrés, est utilisé pour déterminer les paramètres qui permettent d'approcher au mieux les données expérimentales recueillies dans la littérature. L'équation est basée sur l'hypothèse que la forme de la courbe caractéristique sol-eau dépend de la répartition de la taille des pores du sol (à savoir que la perte de saturation est une fonction de cette répartition). L'équation a la forme d'une intégrale de courbe de répartition de fréquences. Cette équation permet un bon ajustement pour les sols sableux, silteux et argileux sur toute la gamme des valeurs de succion, de 0 à $10^6$ kPa.

*Mots clés* : courbe caractéristique sol-eau, répartition de la taille des pores, ajustement non linéaire, succion dans le sol, teneur en eau.

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## Introduction

A theoretical framework for unsaturated soil mechanics has been established over the past two decades. The constitutive equations for volume change, shear strength, and flow for unsaturated soil have become generally accepted in geotechnical engineering (Fredlund and Rahardjo 1993*a*). The measurement of soil parameters for the unsaturated soil constitutive models, however, remains a demanding laboratory process. For most practical problems, it has been found that approximate soil properties are adequate for analysis (Papagiannakis and Fredlund 1984). Hence, empirical procedures to estimate unsaturated soil parameters would be valuable.

Laboratory studies have shown that there is a relationship between the soil-water characteristic curve for a particular soil and the properties of the unsaturated soil (Fredlund and Rahardjo 1993*b*). For example, it has become an acceptable procedure to predict empirically the permeability function for an unsaturated soil by using the saturated coefficient of permeability and the soil-water characteristic curve (Marshall 1958; Mualem 1986; University of Saskatchewan 1984). Similar procedures have been suggested for the shear strength properties of an unsaturated soil (Fredlund and Rahardjo 1993*b*). Since the soil-water characteristic curve is used as the basis for the prediction of other unsaturated soil parameters, such as the permeability and shear-strength functions, it is important to have a reasonably accurate characterization of the soil-water characteristic curve.

This paper reviews the forms of mathematical equations that have been suggested to characterize the soil-water characteristic curve. It appears that none of the suggested equations accurately fit laboratory data over the entire suction range. This paper proposes a new equation that can be used to fit laboratory data over the entire soil suction range. A mathematical basis for the equation is described and a best-fit procedure is outlined to obtain the parameters for the equation.

## Definitions[1]

The soil-water characteristic curve for a soil is defined as the relationship between water content and suction for the soil (Williams 1982). The water content defines the amount of water contained within the pores of the soil. In soil science, volumetric water content $\theta$ is most commonly used. In geotechnical engineering practice, gravimetric water content $w$, which is the ratio of the mass of water to the mass of solids, is most commonly used. The degree of saturation $S$ is another commonly used measure to indicate the percentage of the voids that are filled with water. The above variables have also been used in a normalized form where the water contents are referenced to a residual water content (or to zero water content).

The suction may be either the matric suction (also known as capillary pressure) of the soil (i.e., $u_a - u_w$, where $u_a$ is the pore-air pressure and $u_w$ is the pore-water pressure) or total suction (i.e., matric plus osmotic suction). At high suc-

[1]There are several soil terms that are used interchangeably in the literature. The terminology used in the paper is most consistent with that found in the geotechnical literature. Other terms are used in the geo-environmental, petroleum, and some of the soil science disciplines. Some of these equivalences are as follows: matric suction ≡ capillary pressure, air-entry value ≡ displacement pressure, and soil-water characteristic curve ≡ suction – volumetic water content curve.

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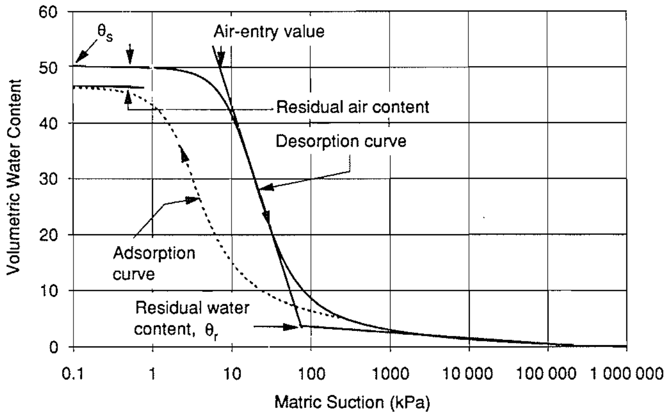


FIG. 1. Typical soil-water characteristic curve for a silty soil.

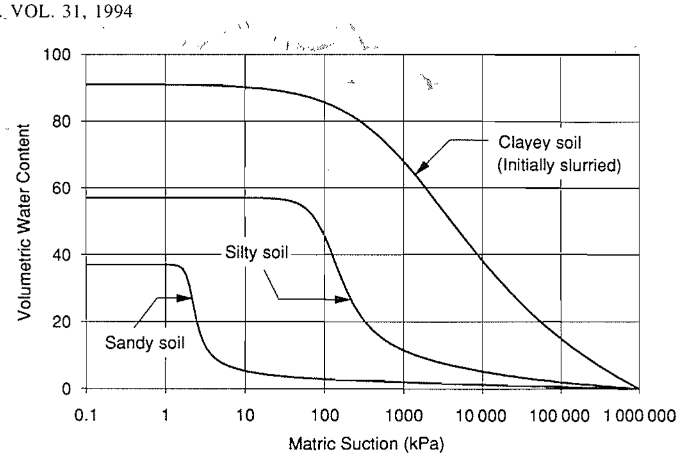


FIG. 2. Soil-water characteristic curves for a sandy soil, a silty soil, and a clayey soil.

tions (i.e., greater than about 1500 kPa), matric suction and total suction can generally be assumed to be equivalent.

As a result of the different terminologies used, the soil-water characteristic curves have taken on numerous forms. It is suggested that the term soil-water characteristic curve be used to represent the relationship between volumetric water content $\theta$ and matric suction. Volumetric water content test results in the low suction range are often presented using an arithmetic scale. Soil-water characteristic curves over the entire suction range are often plotted using a logarithmic scale.

Figure 1 shows a typical plot of a soil-water characteristic curve for a silty soil, along with some of its key characteristics. The air-entry value of the soil (i.e., bubbling pressure) is the matric suction where air starts to enter the largest pores in the soil. The residual water content is the water content where a large suction change is required to remove additional water from the soil. This definition is vague and an empirical procedure for its quantification would be useful. A consistent way to define the residual water content is shown in Fig. 1. A tangent line is drawn from the inflection point. The curve in the high-suction range can be approximated by another line. The residual water content $\theta_r$ can be approximated as the ordinate of the point at which the two lines intersect (Fig. 1). The total suction corresponding to zero water content appears to be essentially the same for all types of soils. A value slightly below $10^6$ kPa has been experimentally supported for a variety of soils (Croney and Coleman 1961). This value is also supported by thermodynamic considerations (Richards 1965). In other words, there is a maximum total suction value corresponding to a zero relative humidity in any porous medium.

The main curve shown in Fig. 1 is a desorption curve. The adsorption curve differs from the desorption curve as a result of hysteresis. The end point of the adsorption curve may differ from the starting point of the desorption curve because of air entrapment in the soil. Both curves have a similar form; however, this paper primarily considers the desorption curve.

Typical soil-water characteristic curves (i.e., desorption curves) for different soils are shown in Fig. 2. The saturated water content $\theta_s$ and the air-entry value or bubbling pressure $(u_a - u_w)_b$, generally increase with the plasticity of the soil. Other factors such as stress history also affect the shape of the soil-water characteristic curves.

## Literature review

Numerous empirical equations have been proposed to simulate the soil-water characteristic curve. Among the earliest is an equation proposed by Brooks and Corey (1964). It is in the form of a power-law relationship:

$$[1] \quad \Theta = \left(\frac{\psi_b}{\psi}\right)^{\lambda}$$

where

$\Theta$ is the normalized (or dimensionless) water content (i.e., $\Theta = (\theta - \theta_r)/(\theta_s - \theta_r)$, where $\theta_s$ and $\theta_r$ are the saturated and residual volumetric water contents, respectively),

$\psi$ is the suction,

$\psi_b$ is the air-entry value, and

$\lambda$ is the pore-size distribution index.

The degree of saturation $S$ has also been used in place of the normalized water content. Equation [1] has been verified through several studies (Campbell 1974; Clapp and Hornberger 1978; Gardner et al. 1970*a*, 1970*b*; Rogowski 1971; Williams et al. 1983; McCuen et al. 1981).

The following linear relationship between the logarithm of volumetric water content and the logarithm of suction was used by Williams et al. (1983) to describe the soil-water characteristic curve of many soils in Australia.

$$[2] \quad \ln \psi = a_1 + b_1 \ln \theta$$

where $a_1$ and $b_1$ are curve-fitting parameters.

McKee and Bumb (1984) suggested an exponential function for the relationship between the normalized water content and suction. This has been referred to as the Boltzmann distribution:

$$[3] \quad \Theta = e^{-(\psi - a_2)/b_2}$$

where $a_2$ and $b_2$ are curve-fitting parameters.

Equations [1] and [3] have been found to be valid for suction values greater than the air-entry value of the soil. The equations are not valid near maximum desaturation or under fully saturated conditions. To remedy this condition, McKee and Bumb (1987) and Bumb (1987) suggested the following relationship:

$$[4] \quad \Theta = \frac{1}{1 + e^{(\psi - a_3)/b_3}}$$

where $a_3$ and $b_3$ are curve-fitting parameters. This equation gives a better approximation in the low-suction range. The equation is not suitable in the high-suction range, since the curve drops exponentially to zero at high suction values.

Equation [1] implies that there is a sharp discontinuity in suction near saturation. Although some coarse-grained sands may have a rapid change in suction at low suctions, most soils, particularly medium- and fine-textured soils, show a gradual curvature in the air-entry region near saturation. A modification of [1] was suggested by Roger and Hornberger (1978) to account for gradual air entry. In the case where the volumetric water content is referenced to zero water content and the normalized volumetric water content $\Theta$ (i.e., $\theta/\theta_s$) is plotted as the abscissa, the general soil-water characteristic plot has an inflection point where the slope $d\psi/d\Theta$ changes from an increasing value to a decreasing value as $\Theta$ decreases. The inflection point is assigned the coordinates ($\Theta_i$, $\psi_i$), and the interval $\Theta_i \le \Theta \le 1$ can be described by a parabola:

[5] $$\psi = -a_4(\Theta - b_4)(\Theta - 1)$$

where $a_4$ and $b_4$ are curve-fitting parameters. The parameters $a_4$ and $b_4$ are obtained by forcing [5] through the two points ($\Theta_i$, $\psi_i$) and (1, 0). The slopes of both [1] and [5] are equal at the inflection point.

Another frequently used form for the relationship between suction and the normalized water content was given by van Genuchten (1980):

[6] $$\Theta = \left[\frac{1}{1+(p\psi)^n}\right]^m$$

where $p$, $n$, and $m$ are three different soil parameters. This form of the equation gives more flexibility than the previous equations described. In an attempt to obtain a closed-form expression for hydraulic conductivity, van Genuchten (1980) related $m$ and $n$ through the equation $m = 1 - 1/n$. This, however, reduces the flexibility of [6]. More accurate results can be obtained by leaving $m$ and $n$ parameters with no fixed relationship.

In 1958, Gardner proposed an equation for the permeability function. The equation emulates the soil-water characteristic curve and can be visualized as a special case of [6]:

[7] $$\Theta = \frac{1}{1+q\psi^n}$$

where:

$q$ is a curve-fitting parameter related to the air-entry value of the soil, and

$n$ is a curve-fitting parameter related to the slope at the inflection point on the soil-water characteristic curve.

## Theoretical basis for the shape of the soil-water characteristic curve

The equations proposed in the research literature are empirical in nature. Each equation appears to apply for a particular group of soils. There are other equations of slightly differing forms that could be tested to assess their fit with experimental data. For example, the soil-water characteristic curve appears to have the form of the right-hand side of a normal-distribution curve. Therefore, the following equation can be used to approximate the soil-water characteristic curve:

[8] $$\Theta = a_5\,e^{-(b_5\psi)^m}$$

where $a_5$, $b_5$, and $m$ are curve-fitting parameters. Equation [8] is not suitable as a general form, although it might apply for some soils over a limited range of suction values.

To establish a theoretical basis for the soil-water characteristic curve, let us consider the pore-size distribution curve for the soil. The soil may be regarded as a set of interconnected pores that are randomly distributed. The pores are characterized by a pore radius $r$ and described by a function $f(r)$, where $f(r)\,dr$ is the relative volume of pores of radius $r$ to $r + dr$. In other words, $f(r)$ is the density of pore volume corresponding to radius $r$. Since $f(r)\,dr$ is the contribution of the pores of radius $r$ to $r + dr$ that are filled with water, the volumetric water content can be expressed as

[9] $$\theta(R) = \int_{R_{min}}^{R} f(r)\,dr$$

where

$\theta(R)$ is volumetric water content when all the pores with radius less than or equal to $R$ are filled with water, and

$R_{min}$ is minimum pore radius in the soil.

Let $R_{max}$ denote the maximum pore radius. Then, for the saturated case,

[10] $$\theta(R_{max}) = \theta_s$$

The capillary law states that there is an inverse relationship between matric suction and the radius of curvature of the air–water interface. In other words, the air–water interface bears an inverse relationship to the pore size being desaturated at a particular suction:

[11] $$r = \frac{C}{\psi}$$

where $C = 2T\cos\varphi$, a constant, where $T$ is surface tension of water, and $\varphi$ is angle of contact between water and soil. Two particular suction conditions can be defined as follows:

[12] $$\psi_{max} = \frac{C}{R_{min}}$$

and

[13] $$\psi_{aev} = \frac{C}{R_{max}}$$

where

$\psi_{max}$ is the suction value corresponding to the minimum pore radius, and

$\psi_{aev}$ is the air-entry suction value of the soil.

Using the capillary law, [9] can be expressed in terms of suction:

[14] $$\theta(\psi) = \int_{\psi_{max}}^{\psi} f\left(\frac{C}{h}\right)d\left(\frac{C}{h}\right) = \int_{\psi}^{\psi_{max}} f\left(\frac{C}{h}\right)\frac{C}{h^2}\,dh$$

where $h$ is a dummy variable of integration representing suction. Equation [14] is the general form describing the relationship between volumetric water content and suction. If the pore-size distribution $f(r)$ of a soil is known, the soil-water characteristic curve can be uniquely determined by [14]. Several special cases are as follows.

(1) *Case of a constant pore size function* — The pore sizes are uniformly distributed, that is, $f(r) = A$, where $A$ is a constant. It follows, from [14], that

[15] $$\theta(\psi) = \int_{\psi}^{\psi_{max}} \frac{AC}{h^2}\,dh = AC\left(\frac{1}{\psi} - \frac{1}{\psi_{max}}\right) = \frac{B}{\psi} - D$$

where $B = AC$, a constant, and $D = AC/\psi_{max}$, a constant.

(2) *Case where pore-size function varies inversely as* $r^2$ — For the case of $f(r) = A/r^2$, the relationship between volumetric water content and suction is

$$[16] \quad \theta(\psi) = \int_{\psi}^{\psi_{max}} \frac{Ah^2}{C^2}\frac{C}{h^2}\,dh = B - D\psi$$

where $B = A\psi_{max}/C$, a constant, and $D = A/C$, a constant. Equation [16] represents a linear variation in the pore sizes. In other words, there is a linear relationship between volumetric water content and suction.

(3) *Case where pore-size function varies inversely as* $r^{(m+1)}$ — For the case of $f(r) = A/r^{(m+1)}$, where $m$ is an integer, the relationship between volumetric water content and suction is

$$[17] \quad \theta(\psi) = \int_{\psi}^{\psi_{max}} \frac{Ah^{m+1}}{C^{m+1}}\frac{C}{h^2}\,dh = B - D\psi^m$$

where $B = A(\psi_{max})^m/(mC^m)$, a constant, and $D = A/(mC^m)$, a constant. The power-law relationship (i.e., eq. [1]) proposed by Brooks and Corey (1964) is simply a special case of [17]. In other words, the Brooks and Corey (1964) power-law relationship is valid only when the pore-size distribution is close to the distribution $f(r) = A/r^{m+1}$.

To describe the soil-water characteristic curve over the entire suction range from 0 to $10^6$ kPa, volumetric water content is referenced to zero water content (otherwise, the normalized water content becomes negative if $\theta$ is less than $\theta_r$). In this case, the normalized water content $\Theta$ becomes $\theta/\theta_s$. Equation [14] suggests that the following integration form can be used as a general form to approximate the soil-water characteristic curve:

$$[18] \quad \theta(\psi) = \theta_s \int_{\psi}^{\infty} f(h)\,dh$$

where $f(h)$ is the pore-size distribution as a function of suction. Equation [18] will generally produce a nonsymmetrical S-shaped curve. Several special cases are as follows.

(1) *Case of a normal distribution*

Let us assume that $f(h)$ is a normal distribution. That is,

$$[19] \quad f(h) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(h-\mu)^2/2\sigma^2}$$

where $\mu$ is mean value of the distribution of $f(h)$, and $\sigma$ is standard deviation of the distribution of $f(h)$. The soil-water characteristic curve defined by [18] can be expressed as follows:

$$[20] \quad \theta(\psi) = \theta_s \int_{\psi}^{\infty} f(h)\,dh = \frac{\theta_s}{2}\frac{2}{\sqrt{\pi}} \int_{(\psi-\mu)/\sqrt{2}\sigma}^{\infty} e^{-y^2}\,dy = \frac{\theta_s}{2}\,\mathrm{erfc}\left(\frac{\psi-\mu}{\sqrt{2}\sigma}\right)$$

where

$$\mathrm{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_{x}^{\infty} e^{-y^2}\,dy = 1 - \mathrm{erf}(x) = 1 - \frac{2}{\sqrt{\pi}} \int_{0}^{\infty} e^{-y^2}\,dy$$

erfc($x$) is the complement of the error function erf($x$). Equation [20] describes a symmetrical S-shaped curve. Therefore, if the pore-size distribution of a soil can be approximated by a normal distribution, the soil-water characteristic curve of the soil will be close to a symmetrical S-shaped curve, and [20] can be used as a model to describe this relationship.

The two fitting parameters (i.e., the mean value $\mu$, and the standard deviation $\sigma$) in [20] are related to the air-entry value of the soil and the slope at the inflection point on the soil-water characteristic curve. If the slope at the inflection point is $s$ and the air-entry value is $\psi_{aev}$, then the standard deviation $\sigma$ can be written as

$$[21] \quad \sigma = \frac{\theta_s}{\sqrt{2\pi}\,s}$$

and the mean value $\mu$ can be calculated as

$$[22] \quad \mu = \psi_{aev} + \frac{\theta_s}{2s}$$

(2) *Case of a gamma distribution*

Consider the case of a gamma-type distribution for the function $f(r)$. That is, $f(h)$ takes the following form:

$$[23] \quad f(h) = \begin{cases} \dfrac{h^{\alpha-1} e^{-h/\beta}}{\beta^\alpha \Gamma(\alpha)}, & \alpha, \beta > 0, \quad 0 \le h < \infty \\ 0, & \text{elsewhere} \end{cases}$$

where

$$\Gamma(\alpha) = \int_0^{\infty} h^{\alpha-1} e^{-h}\,dh$$

In this case, the soil-water characteristic curve defined by [18] has a smaller air-entry value, a steeper slope near saturation, and a gentler slope near the residual water content. In the special case when $\alpha$ is an integer, the soil-water characteristic curve defined by [18] becomes

$$[24] \quad \theta(\psi) = \theta_s \int_{\psi}^{\infty} \frac{h^{\alpha-1} e^{-h/\beta}}{\beta^\alpha \Gamma(\alpha)}\,dh = \frac{\theta_s}{\Gamma(\alpha)} \int_{\psi/\beta}^{\infty} h^{\alpha-1} e^{-h}\,dh = \theta_s \sum_{i=0}^{\alpha-1} \frac{\psi^i e^{-\psi/\beta}}{i!\beta^i}$$

For $\alpha = 1$, the gamma distribution becomes an exponential distribution:

$$[25] \quad f(h) = \begin{cases} \dfrac{1}{\beta} e^{-h/\beta}, & \beta > 0, \quad 0 \le h < \infty \\ 0, & \text{elsewhere} \end{cases}$$

and the soil-water characteristic curve defined by [18] can be