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SPLINE INTERPOLATION: A MEANS OF INTERFACING FINITE ELEMENT ANALYSIS AND LIMIT EQUILIBRIUM ANALYSIS

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ABSTRACT

Geotechnical engineers can perform finite element seepage analyses to augment limit equilibrium slope stability analyses. Typically, a seepage analysis is used to provide pore-water pressure data which in turn is used in the calculation of the factor of safety of a slope. A method of interpolation must be implemented within the slope stability computer program that computes pore-water pressure at any specified location on the cross-section. However, depending on the type of interpolation procedure used, the number of known data points which can be used in the interpolations is limited by the available computer memory.

Spline Interpolation is a spatial interpolation procedure which is widely used to map variables from irregularly spaced data. Recent work has shown that Spline Interpolation is well-suited for interpolating pore-water pressure data for slope stability analyses. The objective of this study is to illustrate how the Spline Interpolation technique can be used as an accurate and cost effective means of interfacing a finite element seepage analysis and a limit equilibrium slope stability analysis. The Spline Interpolation method is compared to the linear least squares method and the inverse distance methods on the basis of estimation accuracy and computing time.

INTRODUCTION

The pore-water pressure conditions in a soil have a significant effect on its strength. As a result, geotechnical engineers perform finite element seepage analyses to augment limit equilibrium slope stability analyses. Typically, a seepage analysis is used to provide pore-water pressure data which in turn is used in the calculation of the factor of safety of a slope. When performing a slope stability analysis, the pore-water pressure values are generally required at positions which fall between the nodes of the finite element mesh. As a result, a method of interpolation is required within the slope stability computer program in order to compute pore-water pressure at any specified location on the cross-section. Factors such as estimation accuracy, computer memory requirements and computational speed must be considered when selecting an appropriate interpolation scheme.

A numerical procedure known as Spline Interpolation was incorporated into the slope stability computer program PC-SLOPE as a means of interpolating pore-water pressure (Fredlund and Fredlund, 1987). Prior to the implementation of this technique, PC-SLOPE relied exclusively on a distance weighting technique known as 4-Way Interpolation for its pore-water pressure estimates. A study conducted by Fredlund et al (1988) showed that, on the basis of estimation accuracy and consistency, the Spline Interpolation technique was superior to the 4-Way Interpolation method. However, because the Spline Interpolation method requires significant computer memory, the number of known data points which can be analyzed (on a microcomputer) within a slope stability computer program is somewhat limited. In the case of PC-SLOPE, Spline Interpolation can be used only when 50 or fewer points are designated. When more than 50 points are specified, the 4-Way Interpolation procedure must be used.

The degree of discretization of the finite element meshes used to analyze seepage problems varies. Typically, these meshes contain from 100 to 1000 or more nodes. Until now, the 4-Way Interpolation method was used within PC-SLOPE when a large number of pore-pressure values were input from a finite element seepage analysis. Fredlund and Barbour (1988) showed that the Spline Interpolation method was a superior alternative to the 4-Way Interpolation method. Therefore, it would be desirable to use the Spline Interpolation method when the number of known data points exceeds 50.

This study presents a procedure for reducing a large amount of finite element data to a smaller data set containing 50 or fewer representative points on a regular grid which can then be used to represent the pore-water pressure data for a slope stability analysis (Figure 1). The primary objective of this paper is to

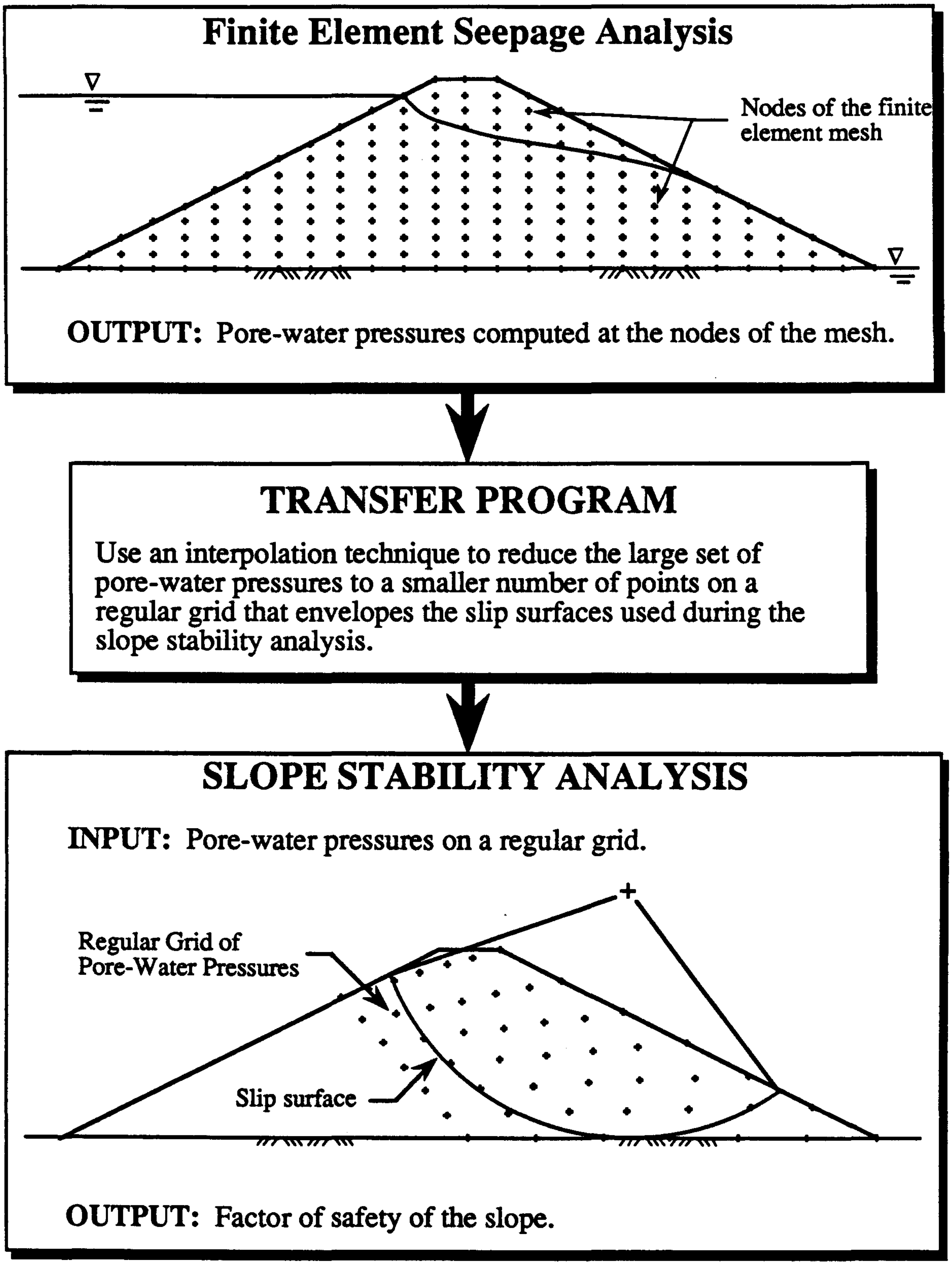

Figure 1 Interfacing Seepage Analysis and Slope Stability Analysis.

present the results of study of the Spline Interpolation technique as an accurate and cost effective means of interfacing finite element data with slope stability analyses. A brief description of the theory associated with Spline Interpolation and comments on its interpolation characteristics are included in this paper. In addition, the results from a comparison of Spline Interpolation with the linear least squares method and the inverse distance methods are presented.

OVERVIEW OF THE SPLINE INTERPOLATION METHOD

Spline Interpolation is numerical procedure that can, given a set of known data points, estimate the value of a parameter at positions where the variable has not been measured or otherwise defined. The objective of this technique is to produce estimates that honor the data points and, at the same time, produce a "smooth" surface or curve. As a result, Spline Interpolation is particularly useful for generating contour maps or wireframe plots from irregularly spaced data .

To illustrate the basic principles of Spline Interpolation it is useful to recall the manual interpolation technique used by draftsmen to draw a smooth curve between a set of one dimensional data points. Thin strips of wood or metal called splines would be used to "fit" a curve to the data points. The draftsman would place weights at various positions along the curve which would fasten the strip. With the strip in place, a smooth continuous line that passed through all data points could then be traced.

For the two dimensional case, one could envision Spline Interpolation as the fitting of a thin plate to a set of two-dimensional coordinates. A function $\sigma(x,y)$ (i.e., an interpolating function) describing the shape of the plate is among all the functions, f, that honor the data points and the one that minimizes a quantity that is analogous to the mechanical energy required to deform a thin strip of infinite extent (Lancaster et al, 1986). The minimized quantity $A(f)$ is equal to (Dubrule, 1983):

$$[1] \quad A(f) = \iint_{R^2} \left[\left(\frac{\partial^2 f}{\partial x^2}\right)^2 + \left(\frac{\partial^2 f}{\partial y^2}\right)^2 + 2\left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 \right]$$

where:

x = coordinate in the x direction
y = coordinate in the y direction
f = all functions that honor the data points

The interpolating function $\sigma(x,y)$ relative to N data points is written as (Duchon, 1975):

$$[2] \quad \sigma(x,y) = a + bx + cy + \sum_{i=1}^{N} \lambda_i h_i^2 \log h_i$$

where a, b, c and λ_i are weighting coefficients and h_i is the distance between two points with coordinates x, y and x_i, y_i, that is:

$$h_i = \sqrt{(x - x_i)^2 + (y - y_i)^2}$$

The coefficients of equation [2] are solved by assembling a system of linear equations where:

$$[3] \quad \begin{cases} \sum_{i=1}^{N} \lambda_i = 0 \\ \sum_{i=1}^{N} \lambda_i x_i = \sum_{i=1}^{N} \lambda_i y_i = 0 \\ \sigma(x_i, y_i) = Z(x_i, y_i) \end{cases}$$

The third condition in equation [3] ensures that the estimated values are equal to the known values at the data points. For a complete description of the mathematical details associated with the Spline Interpolation method, see Ahlberg et al (1967) and Lancaster et al (1986).

From equations [2] and [3] one can see that for N data points the solution of a N+3 by N+3 symmetric matrix is required to determine the weighting coefficients (i.e., a, b, c and λ_i). As a result, the computer memory required to store the matrix (and the time required to solve the matrix) increases as N^2. However, the matrix need only be solved once to compute estimates at any number of points within the domain of the known data points. This becomes an important consideration when repeated estimates are needed in a given region (e.g., pore-water pressure estimates along repeated slip surface trials during a slope stability analysis).

RELATIONSHIP BETWEEN SPLINE INTERPOLATION AND UNIVERSAL KRIGING

Under specific criteria, Spline Interpolation is equivalent to another popular interpolation technique known as Universal Kriging (Matheron, 1981); an estimation technique that has been used extensively in the mining industry. Kriging, like Spline Interpolation, determines a set of weighting coefficients (i.e., called kriging coefficients) from a set of known data points and then is able to predict the value of the parameter at any other location. The Kriging technique determines the coefficients subject to a different criterion; that being the variance of the estimated values from the actual values being minimized. The Kriging technique strives to obtain estimates which are, on average, as close as possible to the actual values. Spline Interpolation, on the other hand, uses the shape of the interpolating function to determine the weighting coefficients which results in estimates that are optimized for aesthetics (i.e., "smoothness") rather than accuracy (Dubrule, 1984).

In order to minimize the estimation variance, it is necessary to know the statistical structure of the variable under study. This involves determining the degree of "drift" in the data (i.e., how the variable changes with respect to its position within the region of interest) and the form of the generalized covariance (i.e., how the value of the parameter varies with respect to distance and direction between data points) (Matheron, 1973). Given N known data points $Z(x_i)$, the kriging estimator $Z^*(x)$ is equal to:

$$[4] \quad Z^*(x) = m(x) + \sum_{i=1}^{N} \lambda_i K(x_i - x)$$

where:

x	= one, two or three-dimensional coordinates
$x_i - x$	= distance between two data points x and x_i
$m(x)$	= drift function
$K(x_i - x)$	= generalized covariance

In two dimensions, the Spline Interpolation technique is equivalent to the Kriging technique when kriging with a linear drift (Matheron, 1981):

$$m(x) = a + bx + cy$$

and a generalized covariance of the form:

$$K(h_i) = h_i^2 \log h_i$$

where:

$$h_i = \sqrt{(x - x_i)^2 + (y - y_i)^2}$$

The Kriging technique requires that the drift and the generalized covariance be determined using a structural analysis. This procedure can be computationally intensive and requires a significant knowledge of geostatistics. Since Spline Interpolation does not require a structural analysis, it can be more easily implemented for use with applications such as slope stability. However, without a structural analysis it is impossible to ascertain how well the chosen drift and covariance functions fit the given data.

OTHER INTERPOLATION TECHNIQUES: TREND SURFACE AND INVERSE DISTANCE METHODS

Linear Trend Surface Method

Engineers often use trend surface analyses to fit a surface to data values. The trend surface analysis is useful for showing broad features of the data. In most cases it is used to remove those broad features in order to allow some other estimation technique (such as Kriging) to work with the residuals (Ripley, 1981). The trend surface is usually a low order polynomial function fit to the data by the least squares technique. In two dimensions, these functions are:

a = a flat function

$a + bx + cy$ = a linear function

$a + bx + cy + dx^2 + exy + fy^2$ = a quadratic function

The linear trend surface function (or drift as we called it in the preceding section) comprises the first portion of the Spline Interpolation estimator (i.e., equation [2]). This is equivalent to passing a sloping two-dimensional plane through the set of data points. Figure 2(a) shows a surface of an aggregate deposit generated from 80 data points using Spline Interpolation while Figure 2(b) shows a linear trend surface generated using the same data. Even though the trend surface is the best least squares fit of the data, this type of plot does not give the analyst a proper feel for local variations of the surface. On the other hand, the plot generated using Spline Interpolation provides a much better perspective for viewing both small and large scale fluctuations.

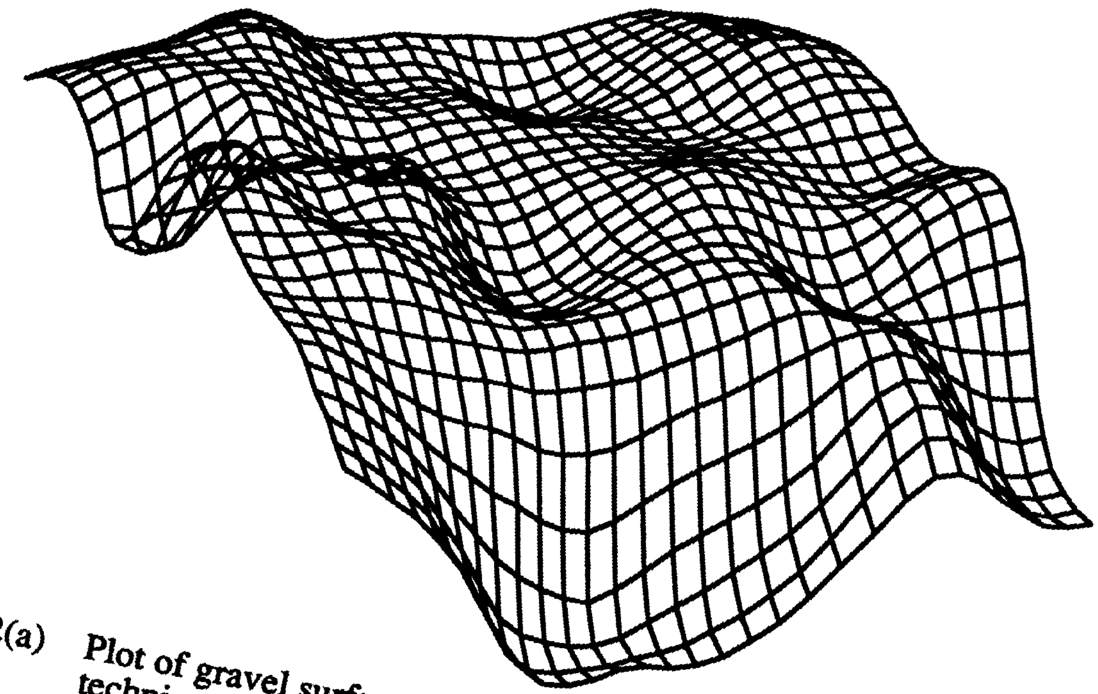

Figure 2(a) Plot of gravel surface generated using the Spline Interpolation technique.

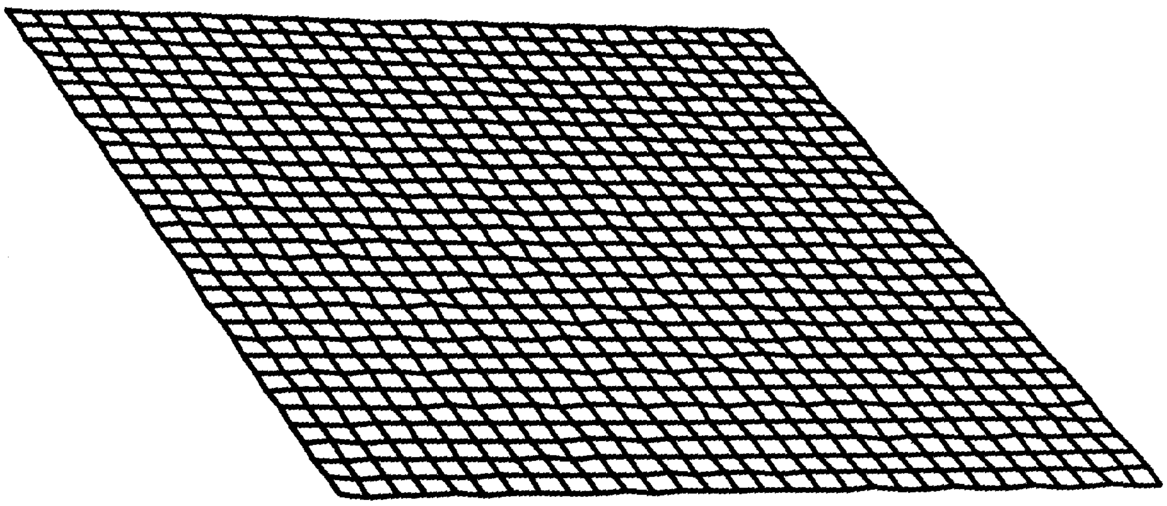

Figure 2(b) Plot of gravel surface generated using the linear least squares method.

Inverse Distance Method

The inverse distance method is a fast and simple interpolation method. Using this technique, the value of the point being estimated is a weighted average of the values at the known data points. The weights are computed as a function of the distances between the point of estimation and the data points:

$$[5] \quad Z^* = \frac{\displaystyle\sum_{i=1}^{n} \frac{Z_i}{d_i^r}}{\displaystyle\sum_{i=1}^{n} \frac{1}{d_i^r}}$$

where:

Z^* = estimated value
Z_i = the value of the neighboring point
d_i = distance to the neighboring point
n = the number of points used in the interpolation
r = the weighting power

The data points are weighted such that the influence of one data point on another declines with distance from the point being estimated. The greater the weighting power, r, the faster the decline in influence and the less effect the data point further away from the point under estimation will have on the computed value. A weighting power of 1 or 2 is used for most applications.

Surfaces produced by the inverse distance method show the major features of the variable but, at the same time, tend to be somewhat irregular. The inverse distance method, unlike the Kriging or the Spline Interpolation methods, is not an exact interpolator (i.e., if the point of estimation coincides with a known data point, the estimated value is not necessarily equal to the measured value). When the point of estimation is "close" to a known data point, the estimate must be forced to equal the measured value. This tends to produce sharp spikes in the surface at the known data points. By comparing the two plots in Figures 3(a) and (b) with those generated from the same data in Figures 2(a) and (b), it can be seen that as the weighting power, r, increases from 1 to 2 the spikes become less pronounced. At the same time, the general shape of the surface becomes more irregular. The inverse distance method is also susceptible to clusters in the data points (Ripley, 1981). For example, a cluster of points near the point of estimation will tend to overshadow any influence of the other points used in the

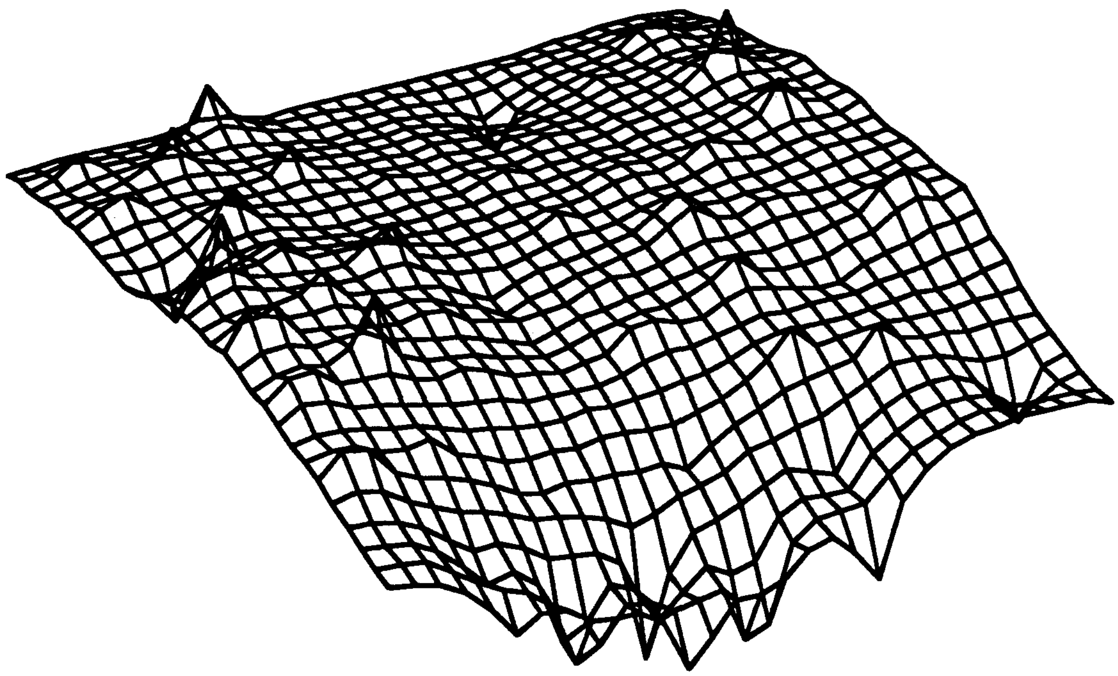

Figure 3(a) Plot of gravel surface generated using the inverse distance method.

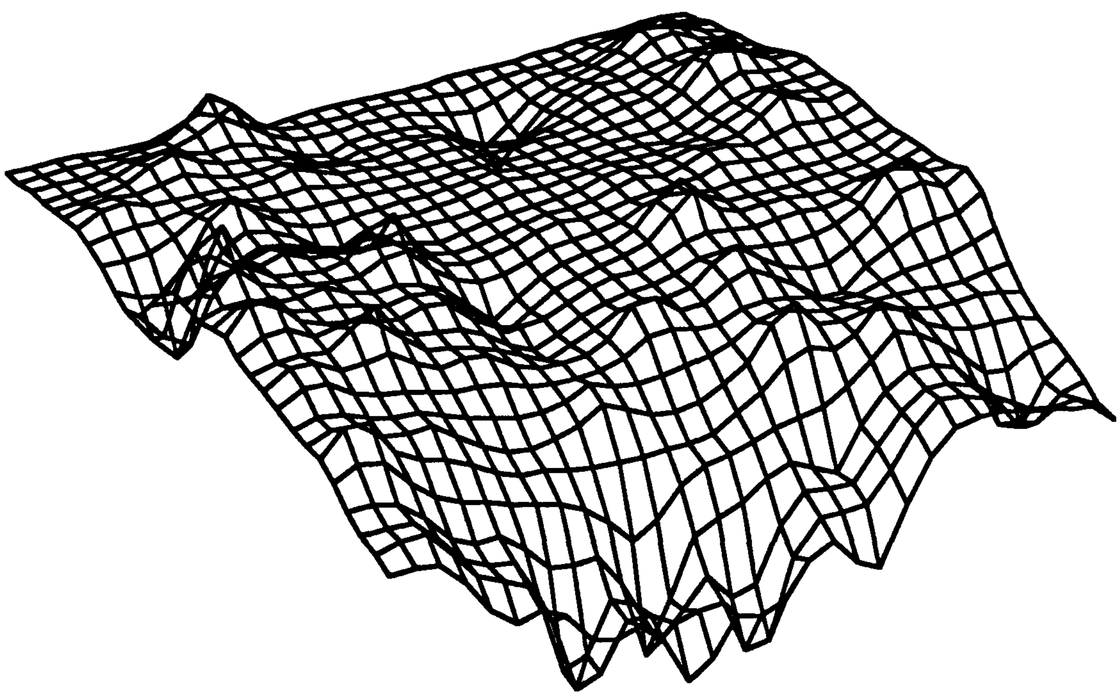

Figure 3(b) Plot of gravel surface generated using the inverse distance squared method.

interpolation. In addition, a surface generated from a set of points that lie on an inclined plane will tend not to look anything like a plane. However, in spite of its drawbacks, the inverse distance method is adequate for many engineering applications.

EVALUATION OF THE INTERPOLATION METHODS

A benchmark example is studied to evaluate the Spline Interpolation, the linear least squares, the inverse distance and the inverse distance squared methods as a means of interfacing finite element seepage analysis data and limit equilibrium slope stability analyses. All interpolation methods are used to generate a rectangular grid of representative pore-water pressure heads over a section of the finite element mesh that envelopes the critical slip surface region. The grid of estimates is then used within the slope stability analysis to compute the factor of safety of the critical slip surface. With these results and the knowledge of an accurate factor of safety for the slope, the interpolation methods are compared on the basis of estimation accuracy. The computational time required to generate the grid of estimates is also used as a basis of comparison.

Description of The Example Problem

An example from the PC-SEEP user's manual (Krahn, 1987) was used to illustrate the transfer of data from a finite element seepage analysis to a slope stability analysis. This example illustrates steady state seepage through a homogeneous earth dam with an impervious lower boundary (Figure 4). The dam cross-section was modelled with a uniform mesh containing 195 nodes and 336 elements. Pore-water pressure heads were computed at each of the nodes.

A preliminary slope stability analysis was performed to determine an accurate value for the factor of safety for the downstream face of the dam. The dam was assumed to have a homogeneous soil profile consisting of a sandy clay till with an unit weight of 19.33 kN/m^3, an angle of internal friction, Ø', of 29 degrees and a cohesion intercept, c', of 24 kPa.

Numerous slip surface positions were analyzed using all pore-water pressure heads and the conventional 4-Way Interpolation method. Once the critical slip surface was identified, the factor of safety was further refined by using a "piezometric line" to represent the pore-water pressures along the critical slip surface. The factor of safety obtained as a result of this analysis was 1.813. This factor of safety value is considered to be the accurate or "true" value since the piezometric line was considered to be the best possible representation of the pore-water pressures. For purposes of comparison, all subsequent slope stability analyses performed during this study used only the critical slip surface.

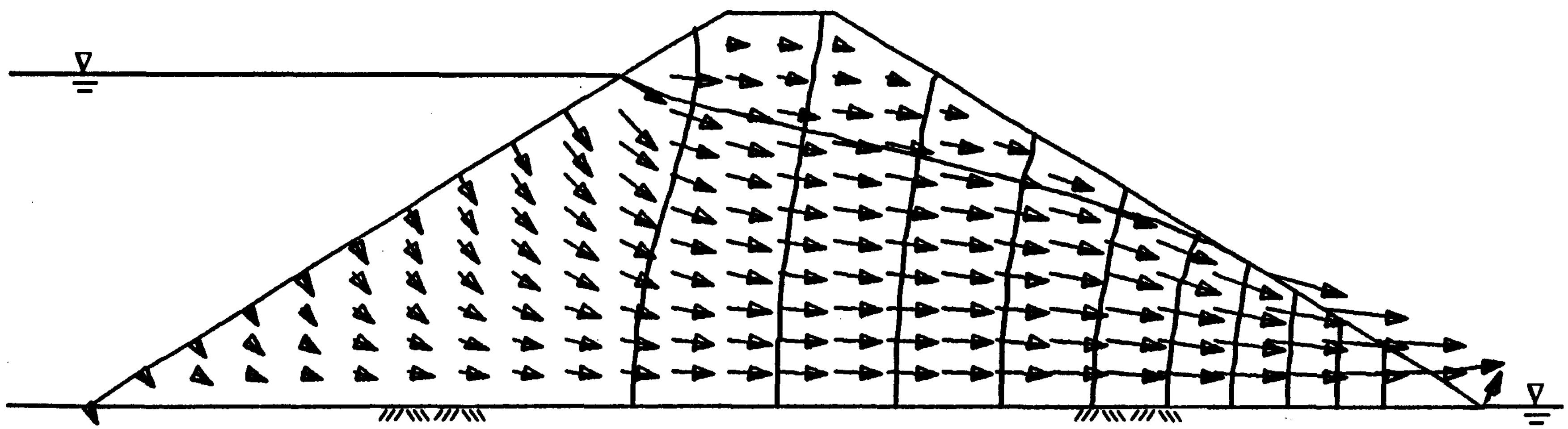

Figure 4 Steady state seepage through a homgenous earth dam.

Computer Programming

A small computer program called T-FER was developed specifically for this study. This program generates a grid of representative pore-water pressure points for any region of the finite element mesh. Spline Interpolation, linear least squares, inverse distance and inverse distance squared methods have been incorporated into the program. Interpolations can be performed using all of the available nodal heads or only those within a specified search radius. Also, by specifying a maximum number of nearest points, n, only the nearest points within the search radius are used in the interpolations. T-FER read nodal head values directly from the PC-SEEP file and wrote the grid of pore-water pressure estimates to a standard ASCII file. These results were then merged into a PC-SLOPE file for subsequent analyses.

Factor of Safety Results

All interpolation methods were used to compute pore-water pressure heads at the nodes of the same grid. In other words, the size and positioning of the grid of estimates were consistent for all analyses. The positioning of the 7 by 7 grid was such that it enveloped the region where the critical surface passed through the cross-section of the dam (Figure 5). The size of the grid chosen maximized the number of data points that can be used (i.e., when using the Spline Interpolation option) within PC-SLOPE.

The interpolation methods were analyzed for their sensitivity to the number of data points used in the interpolations. Each interpolation method was used to generate a grid of estimates using all data points in the finite element mesh (i.e., 195 points) and using the nearest 50, 30, 15, 10 and 5 data points. After each grid of estimates was computed, the generated data was used in two slope stability runs; one run to compute the factor of safety while using the Spline Interpolation method and one run to compute the factor of safety while using the 4-Way Interpolation method. All of the slope stability runs were configured identically to the preliminary slope stability analysis (i.e., single slip surface, same geometry and soil conditions, etc.).

The results of the slope stability runs are shown in Tables 1 and 2. Table 1 presents the factors of safety while using the Spline Interpolation method to interpolate pore-water pressures during the slope stability runs while Table 2 summarizes the factors of safety when the 4-Way Interpolation method is used. These results are presented graphically in Figures 6 and 7. All factors of safety were computed using the Bishop Simplified method.

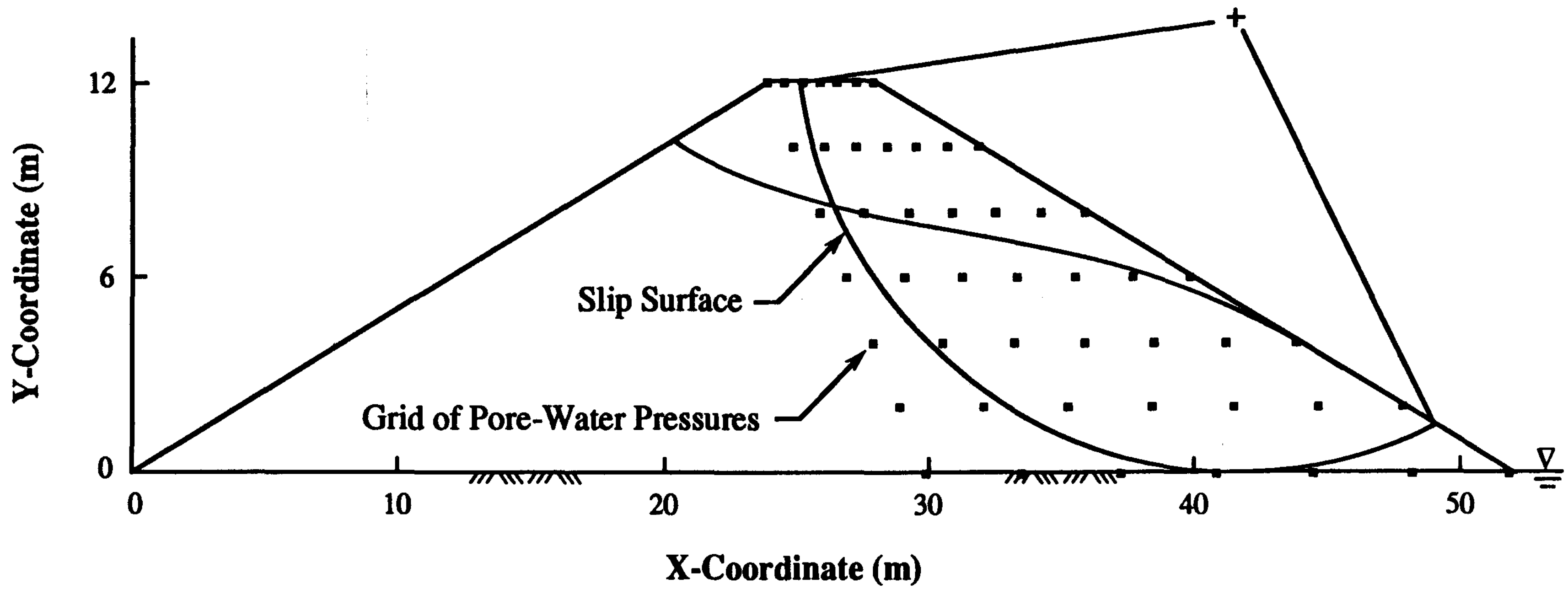

Figure 5 Grid of pore-water pressures superimposed on the cross-section of the dam.

Table 1:
Summary of The Factors of Safety
When Using The Spline Interpolation Option

Interpolation Method Used To Generate Grid	No. of Data Points Used In The Interpolations					
	195	50	30	15	10	5
Spline Interpolation	1.811	1.811	1.810	1.811	1.811	1.811
Linear Least Squares	1.773	1.830	1.819	1.812	1.811	1.811
Inverse Distance	1.834	1.928	1.912	1.890	1.868	1.866
Inverse Distance Squared	1.875	1.876	1.866	1.853	1.841	1.840

Table 2:
Summary of The Factors of Safety
When Using The 4-Way Interpolation Option

Interpolation Method Used To Generate Grid	No. of Data Points Used In The Interpolations					
	195	50	30	15	10	5
Spline Interpolation	1.852	1.851	1.850	1.853	1.853	1.853
Linear Least Squares	1.811	1.857	1.858	1.855	1.855	1.854
Inverse Distance	1.856	1.937	1.924	1.908	1.893	1.893
Inverse Distance Squared	1.897	1.896	1.889	1.882	1.874	1.875

The results shown in Table 1 and Figure 6 indicate that when the Spline Interpolation method was used to generate the grid of estimates, the computed factor of safety remained unchanged over the range of interpolation points tested. This consistent value of the factor of safety is, for all practical purposes, equal to the "true" factor of safety. These results illustrate the consistency of the Spline Interpolation method. The results show that for this example there is no significant benefit to using more than 5 data points for interpolation purposes. This can be attributed in a large part to the simple nature of the flow system studied. It is probable that had there been a more complex flow system involved, there would have been significant deviations in the factor of safety with a changing number of interpolation points.

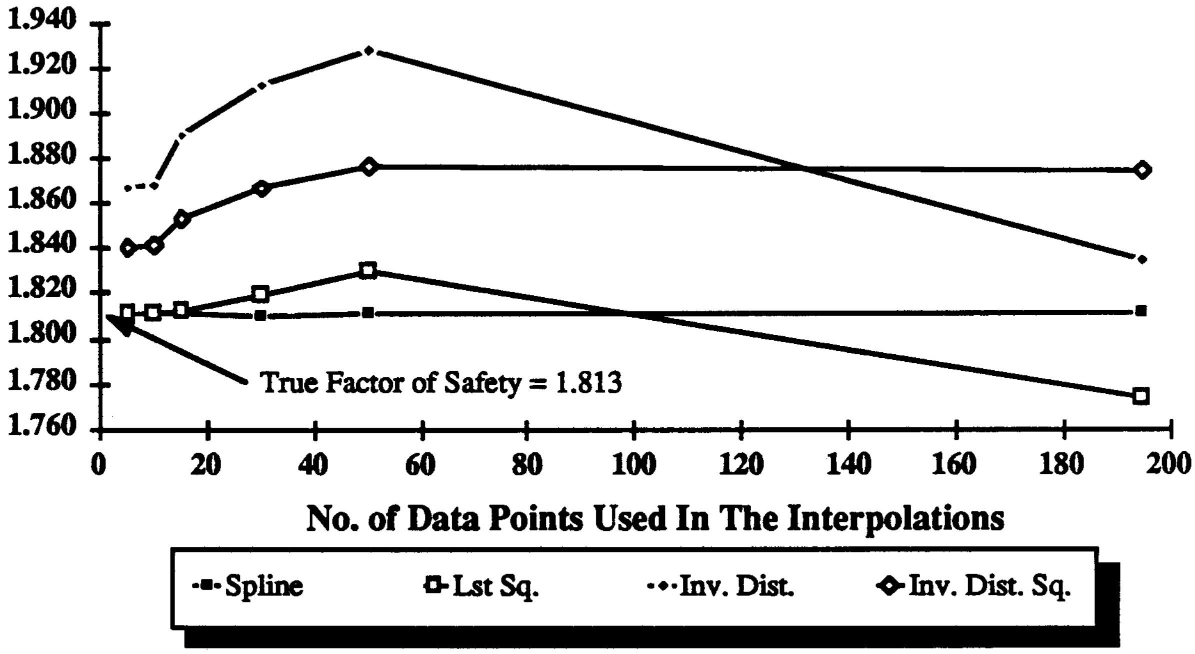

Figure 6 Factors of Safety when using the Spline Interpolation option.

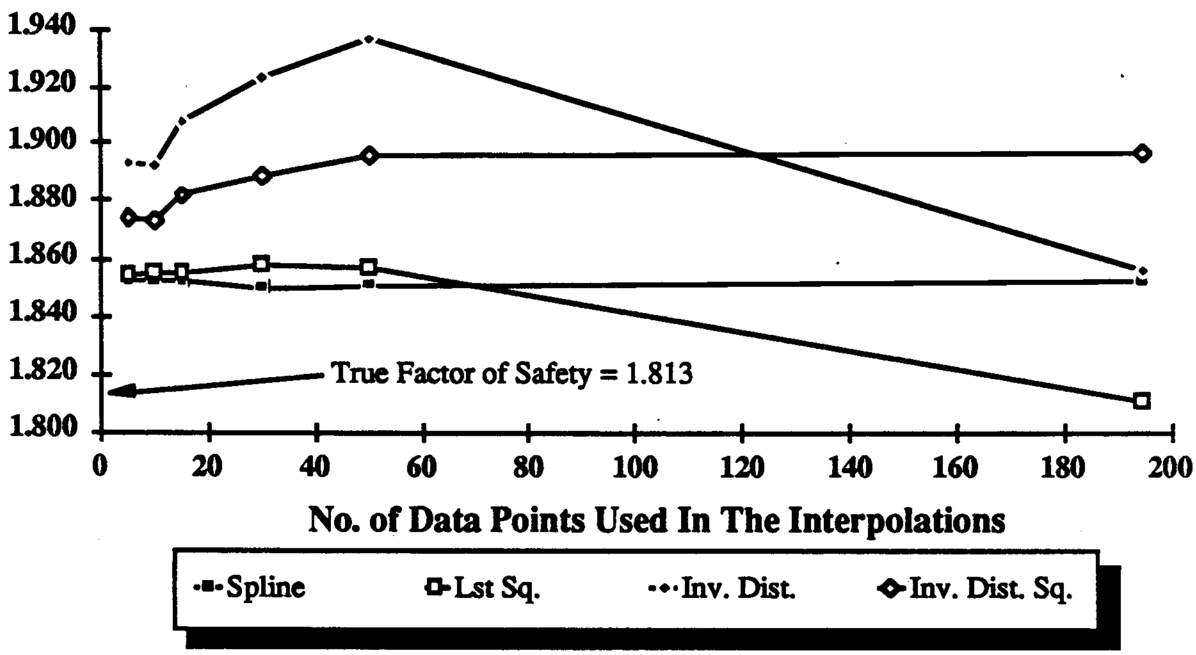

Figure 7 Factors of Safety when using the 4-Way Interpolation option.

The linear least squares method showed favorable factor of safety results when the number of data points used in each interpolation was kept at 15 or lower. In Figure 6, the least squares factor of safety began to deviate from the "true" value when more than 30 points were used. For a neighborhood of data points, the linear least squares estimator is a sloping two dimensional plane. As a result, it would be expected that this sloping plane would be adequate for small neighborhoods but would become a poorer estimate of the true surface as the size of the neighborhood increases. This trend is apparent in the factor of safety results. For example, the computed factor of safety is optimum when the neighborhood of data points is the smallest (i.e., 5 nearest data points) and is the least accurate when the neighborhood is the largest (i.e., all points in the finite element mesh). However, it is probable that a flow problem with large gradients could produce inaccuracies in the least squares method since a planar surface would not be a suitable estimate of the true pore-water pressure surface, even on a small scale. In these cases, the Spline Interpolation method would have a definite advantage over the linear least squares method. The Spline Interpolation method, because of its interpolation properties, is more apt to produce a surface that is a better estimate of the true surface at any scale.

The use of the inverse distance methods to generate grids of representative pore-water pressure heads resulted in the least accurate factor of safety results. The inverse distance method was significantly less accurate than the inverse distance squared method. Both methods showed a tendency to produce poorer estimates as the number of points in the search neighborhood increased which indicates that the interpolation properties of the inverse distance method are not particularly well suited to the spatial properties of the pore-water pressure data.

Figure 7 displays graphically the factors of safety that were computed when using the 4-Way Interpolation method to compute the pore-water pressures along the slip surface from the grids of representative points generated by the four interpolation methods. This exercise verifies the findings of the study conducted by Fredlund et al (1988). In other words, the Spline Interpolation method is superior to the 4-Way Interpolation method on the basis of its estimation accuracy. For almost every type of analysis (i.e., type of interpolation method used to generate the grid and the number of data points used in the interpolations), the use of the Spline Interpolation option in PC-SLOPE resulted in a more accurate factor of safety than when the 4-Way Interpolation option was used. These results also provide justification for the effort that must be expended to interface seepage analyses and slope stability analyses.

When the 4-Way Interpolation method is used to estimate pore-water pressure within a slope stability analysis there is no practical limit to the number of data points that a user can specify. As a result, all pore-water pressure data from seepage analyses can be incorporated directly into the slope stability analysis. However, if it is desirable to achieve the benefits of the superior estimation capabilities of the Spline Interpolation method, it is important to spend some effort on the interfacing procedure.

Comparison of Computational Time

The time required to compute each grid of estimates was measured. A summary of the times is presented in Table 3 and shown graphically in Figure 8. Both inverse distance methods produced identical times and therefore their results have been summarized under one interpolation method. All tests were performed on a 10 MHz IBM XT compatible computer with a math coprocessor.

Table 3
Comparison of Computing Times Required to Generate The Grids of Pore-Water Pressure Heads

	Method Used to Generate Grid*		
No. of Points	Spline	Least Squares	Inverse Distance
195	329	169	13
50	550	361	36
30	183	135	26
15	49	41	12
10	25	23	12
5	16	14	12

*All times in seconds

The Spline Interpolation method and the linear least squares method require significantly more computing time than does the inverse distance methods. Moreover, the computing time increased more rapidly as the number of data points used in the interpolations increased. The least squares method was significantly faster than the Spline Interpolation method when 50 or more data points were used. However, when 30 or fewer neighboring points were used in the interpolations, the computational time required by both methods was essentially the same. All interpolation methods showed a marked drop in accuracy when all pore-water pressure points were used. In the case of the spline and least squares methods, this is a result of only having to solve for one large set of weighting coefficients as opposed to solving for 49 smaller sets. When all the data points are used in the inverse distance interpolations, the search for neighboring points is not required. As a result, a significant saving in computer time is realized.

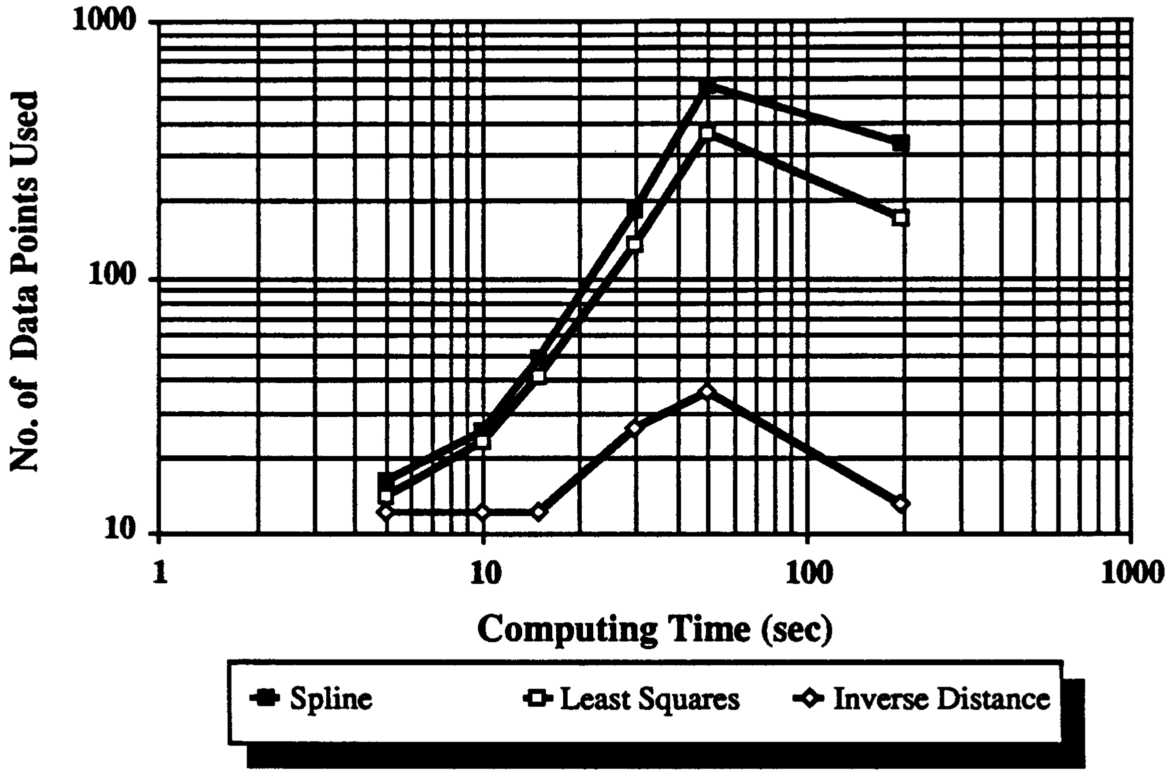

Figure 8 Comparison of computing times by the various interpolation techniques.

CONCLUDING REMARKS

The results of the study show that, on the basis of estimation accuracy and consistency, the inverse distance techniques are inferior to the Spline Interpolation and the linear least squares methods for interfacing seepage and slope stability analyses. The computing time advantage of the inverse distance methods does not offset the less accurate pore-water pressure estimations.

The Spline Interpolation and the linear least squares methods showed comparable results, both in terms of estimation accuracy and computational time. However, the linear least squares method did show a tendency to error as the number of data points used in the interpolations increased. This effect may become more pronounced when pore pressure changes are large and abrupt. It would be desirable to study other more complex flow problems to ascertain whether this is always the case. Based on the consistency of the pore-water pressure estimates over the entire range of data points tested, it can be seen that the Spline Interpolation method is the best interpolation method for interfacing finite element seepage analyses and limit equilibrium slope stability analyses.

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