Non-linearity of strength envelope for unsaturated soils

D.G. Fredlund, H. Rahardjo, and J.K.-M. Gan

Abstract: A literature review on the theory and measurements of shear strength in unsaturated soils is presented. Recent experimental data exhibiting non-linearity in the failure envelope are reported and discussed. The results of direct shear tests on compacted glacial till specimens are presented. Various procedures for handling the non-linear failure envelope are proposed in this paper.

Key words: shear strength, matric suction, direct shear test, non-linearity.

Introduction

The shear strength theory for unsaturated soils has received increased attention during the past three decades. Measurements of relevant shear strength parameters have been made in many parts of the world. The shear strength behaviour of an expansive soil is similar to that of an unsaturated soil due to the presence of negative pore-water pressures. This paper first presents a brief review of relevant literature on the shear strength behaviour of unsaturated soils and traces the gradual evolution of our understanding.

Research during the past few years has revealed that there may be significant non-linearity in the shear strength envelope for unsaturated soils. The nature of this non-linearity is addressed along with a description of possible ways to accommodate it in practice. The manner in which the proposed shear strength equation can be used to describe the non-linear failure envelope is discussed later in this paper. This paper also attempts to address the question of whether all testing procedures will yield unique shear strength parameters.

Review of theory and relevant laboratory data

In 1978, a shear strength equation for an unsaturated soil was proposed using the independent stress state variables (Fredlund et al. 1978). The two stress state variables are the net normal stress, $(\sigma - u_a)$, and the matric suction, $(u_a - u_w)$, where σ is the total normal stress; u_a is the pore-air pressure; and u_w is the pore-water pressure. The uniqueness of the proposed equation was justified using the analysis of triaxial tests results on unsaturated soils published by other researchers.

Several triaxial test techniques for measuring the shear strength of an unsaturated soil were proposed by Bishop et al. (1960). Bishop and Donald (1961) tested an unsaturated loose silt using a consolidated drained test. The axis-translation technique proposed by Hilf (1956) is commonly used for testing unsaturated soils at matric suctions higher than 101 kPa. The high air entry disk must have an air entry value greater than the matric suction of the soil. The acceptability of the axis-translation technique was verified by Bishop and Blight (1963).

Two sets of shear strength test data were reported by Bishop et al. (1960). These tests were on a compacted shale and a compacted boulder clay. The data were later used in the examination of the shear strength equation. proposed for an unsaturated soil by Fredlund et al. (1978). The data were re-analyzed in terms of the two independent stress state variables. The results indicated essentially a planar failure envelope when the shear stress, τ, was plotted with respect to the $(\sigma - u_a)$ and $(u_a - u_w)$ variables.

The results of consolidated drained and constant water content tests on unsaturated Dhanauri clay were presented by Satija (1978) and Gulhati and Satija (1981). The reanalysis of these results appears to support a planar type of failure envelope (Ho and Fredlund 1982; Fredlund and Rahardjo 1987).

A series of consolidated drained direct shear and triaxial tests on unsaturated Madrid gray clay were reported by Escario (1980). These data were further studied by Ho and Fredlund (1982) and showed that a planar failure envelope adequately defined failure conditions. Ho and Fredlund (1982) published the results of multistage, consolidated drained triaxial tests on undisturbed specimens of a decomposed rhyolite and a decomposed granite. The plotted test results showed some non-linearity, but this was interpreted as being related to the nature of the multistage test.

D.G. Fredlund. Professor, Department of Civil Engineering, University of Saskatchewan, 57 Campus Drive, Saskatoon, SK, Canada S7N 5A9.
H. Rahardjo. Research Engineer, Department of Civil Engineering, University of Saskatchewan, 57 Campus Drive, Saskatoon, SK, Canada S7N 5A9.
J.K.-M. Gan, Geotechnical Engineer, Minconsult Sdn. Bhd., 14 Jalan 20/16A, Paramound Carder, 46300 Petaling Jaya Selangor, Malaysia.

Reproduced with permission from the *Proceedings, 6th International Conference on Expansive Soils*, New Delhi, India, December 1–4, 1987. pp. 49–54.

Experimental data analysed in accordance with the shear strength equation proposed by Fredlund et al. (1978) have been shown to indicate essentially a planar failure envelope. However, recent direct shear tests on an unsaturated glacial till has shown a significant non-linearity (i.e., curvature) in the failure envelope with respect to the $(u_a - u_w)$ axis (Gan 1986). Escario and Sáez (1986) also reported a non-linearity in the shear stress, τ, versus matric suction, $(u_a - u_w)$, relationship. Their results were obtained using direct shear tests on three soils.

Theory of shear strength

The proposed shear strength equation (Fredlund et al. 1978) for an unsaturated soil has the following form:

[1] $$\tau_{ff} = c' + (\sigma_{ff} - u_{af}) \tan \phi' + (u_a - u_w)_f \tan \phi^b$$

where:

τ_{ff} = shear stress on the failure plane at failure,

c' = intercept of the "extended" Mohr-Coulomb failure envelope on the shear stress axis when the net normal stress and the matric suction at failure are equal to zero. It is also referred to as the "effective cohesion",

$(\sigma_{ff} - u_{af})$ = net normal stress on the failure plane at failure,

σ_{ff} = total normal stress on the failure plane at failure,

u_{af} = pore-air pressure at failure,

ϕ' = angle of internal friction associated with the net normal stress state variable $(\sigma_{ff} - u_{af})$,

$(u_a - u_w)_f$ = matric suction at failure,

u_{wf} = pore-water pressure at failure, and

ϕ^b = angle indicating the rate of change in shear strength relative to changes in matric suction, $(u_a - u_w)_f$.

Equation [1] describes a planar surface which is called the extended Mohr-Coulomb failure envelope as illustrated in Fig. 1. The surface is tangent to the Mohr circles at failure. The shear strength of an unsaturated soil is considered to consist of an effective cohesion, c', and the independent contributions from net normal stress, $(\sigma - u_a)$, and matric suction $(u_a - u_w)$. The shear strength contribution from net normal stress and matric suction are characterized by ϕ' and ϕ^b angles, respectively.

Experimental evidence indicating non-linearity of the failure envelope

The planar failure envelope as shown in Fig. 1 has been used to describe the shear strength of an unsaturated soil. For a planar failure envelope, the slope angles, ϕ' and ϕ^b, are assumed to be constants. As an example consider the triaxial test results on two compacted soils presented in Fig. 2. The results are plotted on the τ versus $(u_a - u_w)$ plane corresponding to a zero net normal stress at failure (i.e., $(\sigma_{ff} - u_{af})$ is equal to 0). The failure envelopes drawn through the plotted data indicate essentially planar surfaces with respect to the $(u_a - u_w)$ axis. In other words, the ϕ^b angles are constant within the matric suction range used in the experiments (i.e., 0 to 200 kPa).

Fig. 1. Extended Mohr-Coulomb failure envelope.

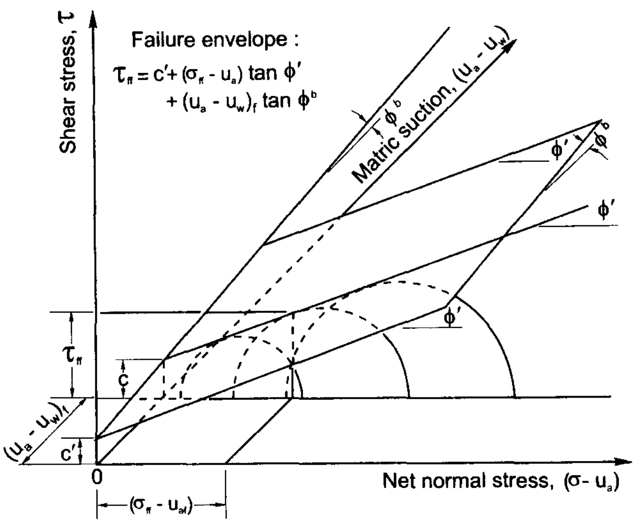
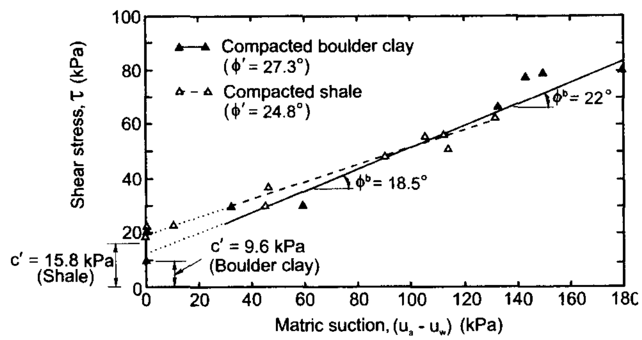


Fig. 2. Planar failure envelopes on the τ versus $(u_a - u_w)$ plane for two compacted soils (data from Bishop et al. 1960).



Recent experimental evidence using a higher range of matric suction (i.e., 0 to 500 kPa) indicates a significant non-linearity in the failure envelope with respect to the matric suction axis. The results of several multistage direct shear tests on a glacial till (Gan 1986) exhibit non-linear failure envelopes (Fig. 3). The failure envelopes are plotted on the τ versus $(u_a - u_w)$ plane corresponding to an average net normal stress at failure of 72.2 kPa. The results fall within a band of failure envelopes as indicated in Fig. 3. The spread in the failure envelopes appears to be due to slight differences in the initial void ratios or initial densities of the soil specimens.

The slopes of the failure envelopes with respect to the $(u_a - u_w)$ axis (i.e., ϕ^b angle) commence at an angle equal to ϕ' (i.e., 25.5°) near saturation and significantly decrease at matric suctions in the range or 50 kPa to 100 kPa. The ϕ^b angles reach a fairly constant value ranging from 5° tc 10° when the matric suction exceeds 250 kPa. Figure 4

Fig. 3. Curved failure envelopes on the τ versus $(u_a - u_w)$ plane for a glacial till (from Gan 1986).

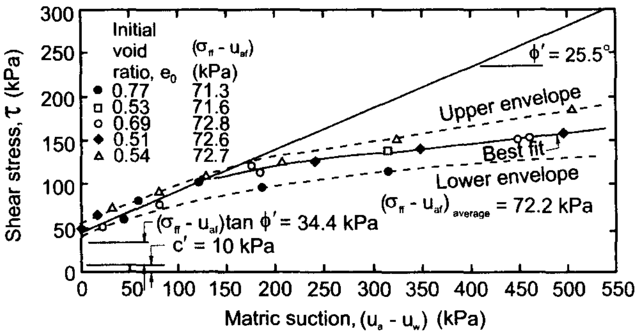


Fig. 4. ϕ^b values corresponding to the three curved failure envelopes (from Gan 1986).

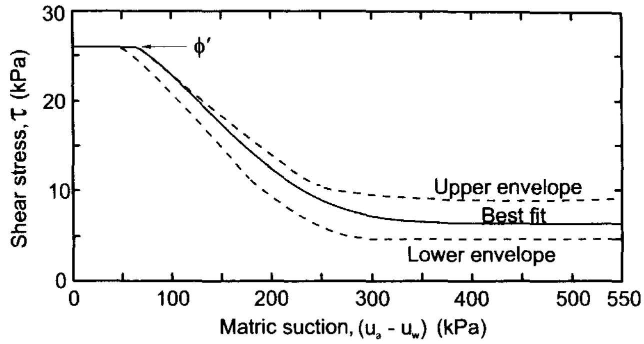


Table 1. Triaxial tests on compacted Dhanauri clay (data from Satija 1978).

Soils	CU tests on *saturated* specimens		Analysis of tests results on *unsaturated* specimens performed by Ho and Fredlund (1982)		
	c' (kPa)	ϕ' (°)	Types of tests	c' (kPa)	ϕ^b (°)
Low density, ρ_d = 1478 kg/m^3, w = 22.2%	7.8	29	CD	20.3	12.6
			CW	11.3	16.5
High density, ρ_d = 1580 kg/m^3, w = 22.2%	7.8	28.5	CD	37.3	16.2
			CW	15.5	22.6

Note: ρ_d = dry density; w = water content; CU = consolidated undrained; CD = consolidated drained; CW = constant water content.

shows the variation in the ϕ^b angle with respect to matric suction for the three failure envelopes.

An analysis of the triaxial test results on compacted Dhanauri clay also reveals some non-linearity in the failure envelope. The original shear strength parameters along with the parameters computed by Ho and Fredlund (1982), using a planar failure envelope are summarized in Table 1. The linear interpretation of the failure envelope results in different c' and ϕ^b values for the same soil tested under two different types of tests (i.e., consolidated drained and constant water content tests). This means that different types of tests on the same soil yield different shear strength parameters.

In other words, the use of a planar failure envelope in analyzing these data causes a problem of non-uniqueness in the resulting shear strength parameters. In addition, the c' values obtained from the analysis do not agree with values obtained from triaxial tests on saturated specimens (Table 1).

The problem of non-uniqueness in the failure envelope necessitates a re-evaluation of the shear strength data presented by Satija (1978). The reanalysis of these data was performed by assuming a curved failure envelope with respect to the $(u_a - u_w)$ axis. The results are plotted on the τ versus $(u_a - u_w)$ plane corresponding to a zero net normal stress at failure (i.e., $(\sigma_{ff} - u_{af})$ equal to 0). Figures 5 and 6 present the results for compacted Dhanauri clay at low and high densities, respectively. The shear strength parameters, c' and ϕ' obtained from the consolidated undrained tests on the saturated specimens (Table 1) were used in this reanalysis. The curved failure envelopes have a cohesion intercept of c' and a slope angle, ϕ^b, equal to ϕ' starting at zero matric suction. The ϕ^b angles start to decrease significantly at matric suctions less than 50 kPa for the low density specimens and at matric suctions of 75 kPa to 100 kPa for the high density specimens. For the low density specimens, the ϕ^b angles reach a constant value of 11° when the matric suction exceeds 150 kPa (Fig. 5*b*). The ϕ^b angles for the high density specimens reach a constant value of 9° when the matric suction exceeds 300 kPa (Fig. 6*b*).

There is now good agreement between the failure envelopes through the consolidated drained and constant water content test results when using the curved failure envelopes depicted in Figs. 5 and 6. In other words, the application of a curved failure envelope to these data leads to an essentially unique failure envelope for the same soil tested under different stress paths or procedures. The uniqueness of the curved failure envelope is also demonstrated at two densities as shown in Figs. 5 and 6. The soil prepared at each initial density should be considered as an independent soil.

Procedures for accommodating the non-linear failure envelope

A typical non-linear failure envelope with respect to the $(u_a - u_w)$ axis is shown in Fig. 7. The envelope has a cohesion of intercept of c' plus the term $((\sigma_{ff} - u_{af}) \tan \phi')$ which is due to the applied net normal stress at failure

Fig. 5. Non-linearity in the failure envelope with respect to the $(u_a - u_w)$ axis: *a*) curved failure envelopes for compacted Dhanauri clay at low density (data from Satija 1978); *b*) corresponding ϕ^b values.

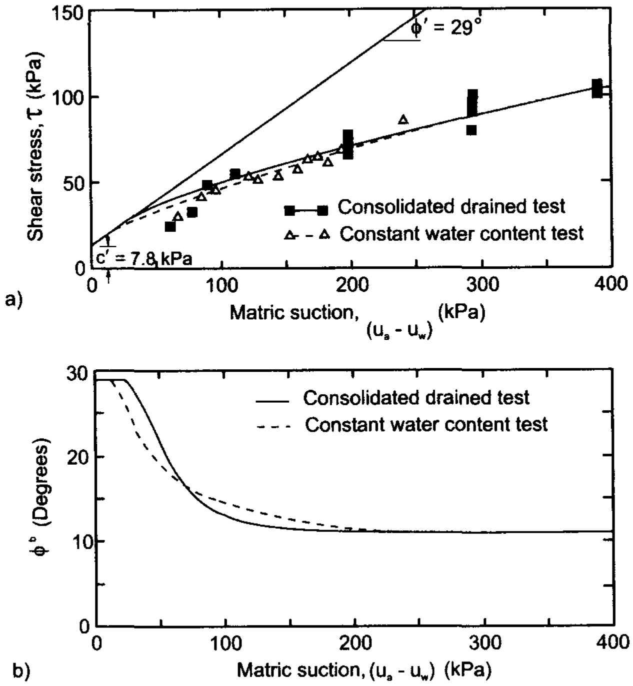
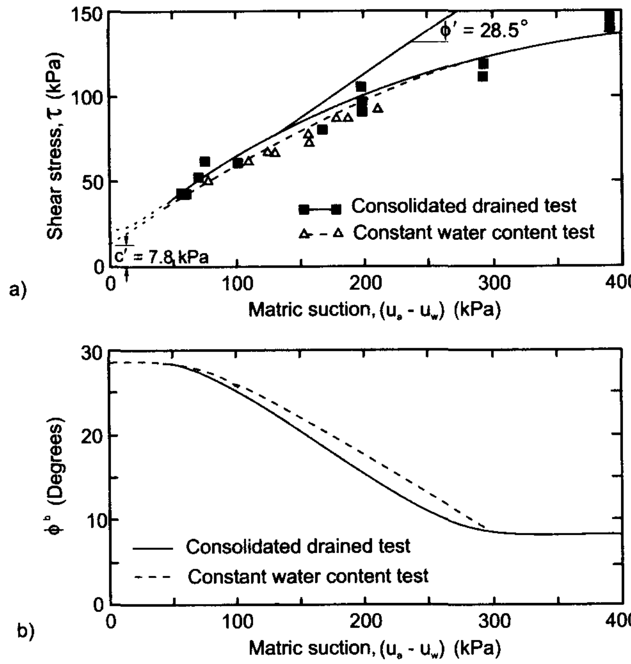


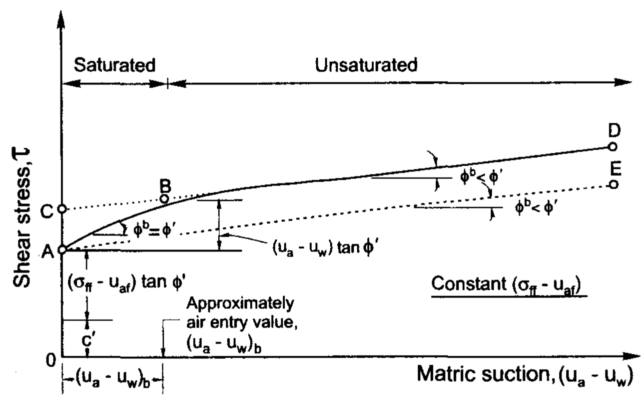
Fig. 6. Non-linearity in the failure envelope with respect to the $(u_a - u_w)$ axis: a) curved failure envelopes for compacted Dhanauri clay at high density (data from Satija 1978); *b*) corresponding ϕ^b values.



(i.e., point A). The ϕ^b angle is equal to ϕ' at low matric suction (i.e., $\overline{AB}$) and decreases to a lower value at high matric suctions (i.e., $\overline{BD}$). The variation in the ϕ^b angle with respect to matric suction can be better understood by considering the pore-volume on which the pore-water pressures act. At low matric suctions, the soil specimen remains saturated and the entire pore-volume is filled with the pore-water. At saturation, the pore-water pressure and the total normal stress are referenced to the external air pressure. In this case, the effects of pore-water pressure and total normal stress on the shear strength are characterized by the same friction angle, ϕ'. Therefore, an increase in matric suction produces the same increase in shear strength as does an increase in the net normal stress. In other words, the ϕ^b angle is equal to ϕ'. Substituting ϕ' for ϕ^b in eq. [1] reduces the equation to that used for a saturated soil. This means that the shear strength equation for saturated soils is applicable when the pore-water pressures are negative, provided the soil remains saturated (Aitchison and Donald 1956).

As matric suction is increased, water is drawn out from the pores of the soil. When the air entry value of the soil, $(u_a - u_w)_b$, is reached, air displaces water in the pores (i.e., desaturation begins). For an unsaturated soil, the pore-water pressure and the total normal stress are referenced to the pore-air pressure as illustrated by the $(u_a - u_w)$ and $(\sigma - u_a)$ variables. The pore-water now occupies only a portion or the soil pores. A further increase in matric suction proves not as effective as an increase in net normal stress in increasing the shear strength of the soil. Figure 7

Fig. 7. Non-linearity in the failure envelope on the τ versus $(u_a - u_w)$ plane.



indicates a decrease in the ϕ^b angle to a value lower than ϕ' as matric suction is increased beyond point *B*. The matric suction corresponding to point *B* appears to be correlative with the air entry value of the soil. The air entry value depends largely on the grain size distribution but may depend to some extent upon the net confining pressure. The effects of normal stress and pore-water pressure on the shear strength of an unsaturated soil are best independently evaluated with reference to the $(\sigma - u_a)$ and $(u_a - u_w)$ stress state variables (i.e., eq. [1]).

Experimental results to date indicate that the soil has a fairly constant value for the ϕ^b angle beyond the breaking point *B* (Fig. 7). Some results appear to exhibit essentially no non-linearity in the failure envelope with respect to the matric suction (see Fig. 2). Non-linearity in the failure envelope can be handled in one of several ways. First the non-linearity illustrated in Fig. 7 can be accommodated using two linear envelopes, $\overline{AB}$ and $\overline{BD}$. The $\overline{AB}$ envelope corresponds to matric suctions less than the soil air entry value, while the $\overline{BD}$ envelope can be used for matric suctions greater than the air entry value. The $\overline{AB}$ and $\overline{BD}$ envelopes have a slope angle of ϕ and ϕ^b respectively. The cohesion intercept of the $\overline{AB}$ and $\overline{BD}$ envelopes are indicated by points A and C, respectively in Fig. 7. A second procedure is to use only one linear envelope with a slope angle equal to ϕ^b starting from the condition of zero matric suction. This case is illustrated by line $\overline{AE}$ in Fig. 7. The shear strength predicted using the $\overline{AE}$ failure envelope will be lower than the actual strength. A third procedure is to discretize the failure envelope into several linear segments with varying ϕ^b angles. This may be necessary in the case of a highly non-linear envelope.

A fourth procedure is to linearize the failure envelope using the method illustrated in Fig. 8. As noted previously, the matric suction along line $\overline{AB}$ in Fig. 7 produces the same increase in shear strength as does the net normal stress. This means that the effect of their changes can be characterized by the same friction angle, ϕ'. Therefore, the line $\overline{AB}$ portion of the failure envelope can be translated onto the τ versus $(u_a - u_w)$ plane (Fig. 8). As the matric suction increases beyond the air entry value or the soil, $(u_a - u_w)_b$, the soil begins to desaturate. At this point the matric suction axis can be used to separate the different effects of normal stress and matric suction on the shear strength. This causes line $\overline{BD}$ to be drawn on the shear stress versus matric suction plane (Figs. 8 and 9). In other words, the curved failure envelope can be linearized if the $(u_a - u_w)$ axis is drawn starting from the air entry value, $(u_a - u_w)_b$, (Figs. 8 and 9), as opposed to starting from zero matric suction (Fig. 7). It is assumed that the air entry value remains constant at various net confining pressures.

Conclusions

The following conclusions can be made:

(1) The relationship between shear strength and matric suction may be somewhat non-linear.

(2) At low matric suctions when the soil remains saturated, the ϕ^b angle is approximately equal to the ϕ' angle. As the matric suction exceeds the air entry value of the soil, desaturation commences and the ϕ^b angle appears to reduce to a relatively constant value. On the basis of existing data, the curved failure envelope can be approximated by a bi-linear envelope.

(3) Various other procedures can be used to accommodate the non-linearity of the failure envelope with respect to matric suction. These procedures are outlined in the text. It is also possible to linearize the failure envelope by translating the portion of the failure envelope corresponding to lower matric suctions, to the τ versus $(\sigma - u_a)$ plane. This procedure eliminates the need for using different shear strength equations for different matric suctions.

Fig. 8. Linearized extended Mohr-Coulomb failure envelope.

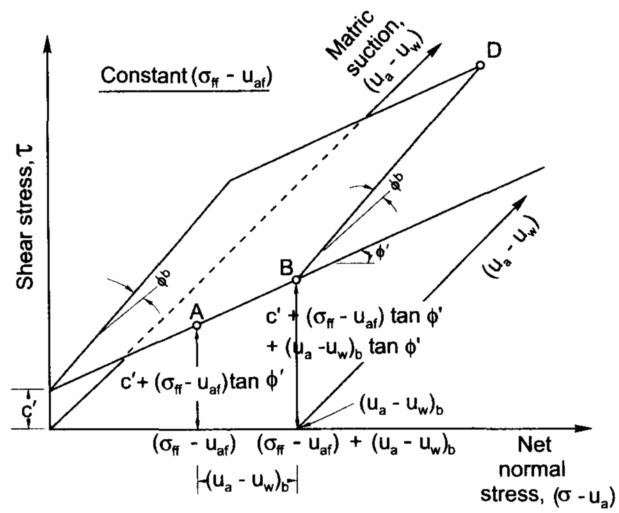
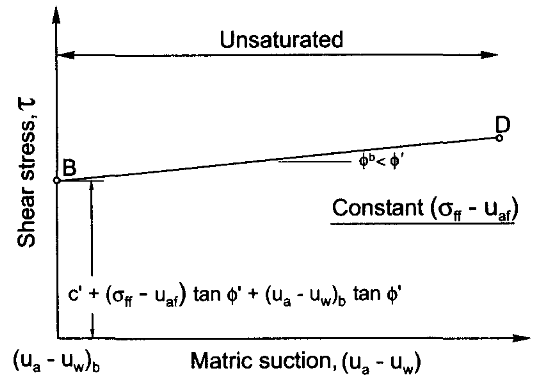


Fig. 9. Linearized failure envelope on the τ versus $(u_a - u_w)$ plane.



References

Aitchison, G.D., and Donald, I.B. 1956. Effective stresses in unsaturated soils. Proceedings, 2nd Australia – New Zealand Conference on Soil Mechanics, pp. 192–199.

Bishop, A.W., Alpan, I., Blight, G.E., and Donald, I.B. 1960. Factors controlling the shear strength of partly saturated cohesive soils. Proceedings, ASCE Research Conference on Shear Strength of Cohesive Soils, University of Colorado, Boulder, CO, pp. 503–532.

Bishop, A.W., and Donald, I.B. 1961. The experimental study of partly saturated soil in the triaxial apparatus. Proceedings, 5th International Conference on Soil Mechanics and Foundation Engineering, Paris, France, Vol. 1, pp. 13–21.

Bishop, A.W., and Blight, G.E. 1963. Some aspects of effective stress in saturated and unsaturated soils. Géotechnique, **13**(3): 177–197.

Escario, V. 1980. Suction controlled penetration and shear tests. Proceedings, 4th International Conference on Expansive Soils, Denver, CO, American Society of Civil Engineers, Vol. 2, pp. 781–797.

Escario, V., and Sáez, J. 1986. The shear strength of partly saturated soils. Géotechnique, **36**(3): 453–456.

Fredlund, D.G., Morgenstern, H.R., and Widger R.A. 1978. The shear strength of unsaturated Soils. Canadian Geotechnical Journal, **15**(3): 313–321.

Fredlund, D.G., and Rahardjo, H. 1987. Soil mechanics principles for highway engineering in arid regions. Transportation Research Record 1137, Washington, D.C., pp. 1–11.

Gan, J. K.-M. 1986. Direct shear strength testing of unsaturated soils. M.Sc. thesis, University of Saskatchewan, Saskatoon, SK, Canada.

Gulhati, S.K., and Satija, B.S. 1981. Shear strength of partially saturated soils. Proceedings, 10th International Conference on Soil Mechanics and Foundation Engineering, Stockholn Sweden, Vol. 1, pp. 609–612.

Hilf, J.W. 1956. An Investigation of pore pressures in com pacted cohesive soils. Technical Memorandum 654, U.S. De partment of Interior, Bureau of Reclamation, Denver, CC USA.

Ho, D.Y.F., and Fredlund, D.G. 1982. The increase in shea strength due to soil suction for two Hong Kong soils. Pro ceedings, ASCE Geotechnical Conference on Engineerin and Construction in Tropical and Residual Soils, Honolul Hawaii, USA.

Satija, B.S. 1978. Shear behavior of partly saturated soils. Ph.D thesis, Indian Institute of Technology, New Delhi, India.