# Unsaturated soil consolidation theory and laboratory experimental data

**D.G. Fredlund and H. Rahardjo**

**Abstract**: The volume change process of an unsaturated soil was considered as a transient flow problem with water and air flowing simultaneously under an applied stress gradient. Laboratory experiments were performed to study the volume change behavior of unsaturated soils. Several compacted kaolin specimens were tested using modified Anteus oedometers and triaxial cells. Theoretical analyses were also made to best-fit the results from laboratory experiments. The fitting was accomplished by approximating the compressibility coefficients and adjusting the coefficients of permeability. The comparisons of the results obtained from the theory and the laboratory yield to a similar behavior of volume changes with respect to time.

*Key words:* unsaturated soil, volume change, consolidation, two-phase flow

## Introduction

For saturated soils, a settlement analysis is usually performed using Terzaghi's theory of consolidation. In practice, many geotechnical problems involve unsaturated soils. The construction of earth-fill dams, highways, and airport runways always use unsaturated compacted soils. In addition, large portions of the earth's surface are covered with residual soils that are unsaturated. Heave and settlement problems are commonly encountered with these unsaturated soils.

The constitutive relations for volume change in an unsaturated soil were proposed by Fredlund and Morgenstern (1976). Laboratory experiments were also performed to study the volume change behavior of unsaturated soils (Fredlund 1973).

This paper presents theoretical analyses of the volume change process in unsaturated soils based on the proposed constitutive relations. Comparisons are made between the theoretical analyses and the laboratory time rate of deformation.

## Theory

A stress gradient applied to an unsaturated soil will cause two phases (i.e., soil particle and contractile skin or air-water interface) to come to equilibrium, whereas the other two phases (i.e., air and water) will flow. The volume change process of an unsaturated soil can be treated as a transient flow problem with water and air flowing simultaneously under an applied stress gradient.

The increase (swelling) or the decrease (consolidation) in the overall volume of an unsaturated soil can be predicted using the proposed constitutive relations (Fredlund and Morgenstern 1976). Assuming the soil particles are incompressible and the volume change of the contractile skin is internal to the element, the continuity requirement for a referential element can be written as

$$[1] \qquad d\varepsilon = d\theta_w + d\theta_a$$

where:

$d\varepsilon$ = change in unit volumetric strain of the soil structure; that is, $d\varepsilon = d\varepsilon_x + d\varepsilon_y + d\varepsilon_z$ where $\varepsilon_x, \varepsilon_y$ and $\varepsilon_z$, are the normal strains in the $x$-, $y$-, and $z$-directions, respectively,

$d\theta_w$ = net inflow or outflow of water from the unit element per unit volume, and

$d\theta_a$ = net inflow or outflow of air from the unit element per unit volume.

Equation [1] shows that only two of the three possible constitutive relations (i.e., the soil structure, water and air phases) are independent (Fredlund 1982). Therefore two constitutive relations are required to describe the volume change behavior in an unsaturated soil. The volume change constitutive relations for unsaturated soils were formulated using two independent stress-state variables, $(\sigma - u_a)$ and $(u_a - u_w)$ (Fredlund 1982), where, $\sigma$ is the total normal stress, $u_a$ represents the pore-air pressure, $u_w$ the pore-water pressure, and $(u_a - u_w)$ the matric suction.

The constitutive equations for the soil structure, water and air phases are, respectively,

$$[2] \qquad d\varepsilon = m_1^s d(\sigma - u_a) + m_2^s d(u_a - u_w)$$

$$[3] \qquad d\theta_w = m_1^w d(\sigma - u_a) + m_2^w d(u_a - u_w)$$

$$[4] \qquad d\theta_a = m_1^a d(\sigma - u_a) + m_2^a d(u_a - u_w)$$

where:

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$m_1^s$ = compressibility of soil structure with respect to a change in $(\sigma - u_a)$,
$m_2^s$ = compressibility of soil structure with respect to a change in $(u_a - u_w)$,
$m_1^w$ = slope of $(\sigma - u_a)$ versus $\theta_w$,
$m_2^w$ = slope of $(u_a - u_w)$ versus $\theta_w$,
$m_1^a$ = slope of $(\sigma - u_a)$ versus $\theta_a$, and
$m_2^a$ = slope of $(u_a - u_w)$ versus $\theta_a$.

Considering the volumetric continuity requirement shown in eq. [1], the following relationship for the compressibilities must be maintained:

$$[5] \qquad m_1^s = m_1^w + m_1^a$$

$$[6] \qquad m_2^s = m_2^w + m_2^a$$

The pore-water and pore-air pressures change with time and can be solved simultaneously using two independent partial differential equations. This method is called a two-phase flow approach and has been presented by Fredlund and Hasan (1979), Lloret and Alonso (1980) and Fredlund (1982).

In this paper, a one-dimensional transient flow process will be considered. The one-dimensional flow equations for the water and air phases can be derived by equating the time derivative of the relevant constitutive equation to the divergence of the velocity as described by the flow laws. Darcy's and Fick's laws are applied to the flow of water and air phases, respectively. The time derivative of the constitutive equation represents the amount of deformation that occurs under various stress conditions, while the divergence of velocity describes the rate of flow of the air and water.

Several assumptions are used in the derivations: (1) isotropic soil, (2) continuous air phase, (3) infinitesimal strains, (4) linear constitutive relations, (5) coefficients of permeability of water and air phases are functions of the volume-mass soil properties during the transient process, and (6) the effects of air diffusing through water, air dissolving in water, and the movement of water vapor are ignored.

The partial differential equation for the water phase in the y-direction can be written as (Fredlund 1982):

$$[7] \qquad \frac{\partial u_w}{\partial t} = -C_w \frac{\partial u_a}{\partial t} + c_v^w \frac{\partial^2 u_w}{\partial y^2} + \frac{c_v^w}{k_w} \frac{\partial k_w}{\partial y} \frac{\partial u_w}{\partial y} + c_g \frac{\partial k_w}{\partial y}$$

where:

$t$ = time,
$C_w = (1 - m_2^w / m_1^w) / (m_2^w / m_1^w)$ and is called the interaction constant associated with the water phase equation,
$c_v^w = k_w / (\rho_w g m_2^w)$ and is called the coefficient of consolidation with respect to the water phase,
$k_w$ = coefficient of permeability with respect to the water phase,
$\rho_w$ = density of water,
$g$ = gravitational acceleration, and
$c_g = 1 / m_2^w$ and is commonly referred to in the soil science literature as the gravity term constant.

The coefficient of permeability of water can vary significantly with space in the unsaturated soil. This variation is taken into account by the last two terms of eq. [7]. The volume-mass properties of the soil can be used to describe the variation in the coefficient of permeability (Corey 1957; Green and Corey 1971).

The air phase partial differential equation has the form (Fredlund 1982):

$$[8] \qquad \frac{\partial u_a}{\partial t} = -C_a \frac{\partial u_w}{\partial t} + c_v^a \frac{\partial^2 u_a}{\partial y^2} + \frac{c_v^a}{D^*} \frac{\partial D^*}{\partial y} \frac{\partial u_a}{\partial y}$$

with:

$$C_a = \frac{m_2^a / m_1^a}{(1 - m_2^a / m_1^a) + \dfrac{(1-S)n}{(u_a - u_{atm}) m_1^a}}$$

and

$$c_v^a = \frac{D^* R\theta}{\omega} \frac{1}{(1 - m_2^a / m_1^a)(u_a + u_{atm}) m_1^a + (1-S)n}$$

where:

$C_a$ = interaction constant associated with the air phase equation,
$S$ = degree of saturation of the soil,
$n$ = porosity of the soil,
$u_{atm}$ = atmospheric pressure,
$c_v^a$ = coefficient of consolidation with respect to the air phase,
$D^* = D / g$, the transmission constant of proportionality for the air phase,
$D$ = a transmission constant for the air phase having the same unit as coefficient of permeability,
$R$ = universal gas constant,
$\theta$ = absolute temperature, and
$\omega$ = molecular weight of the mass of air.

The last term in eq. [8] accounts for the variation of the transmission constant of proportionality for the air phase with respect to space. The constant of proportionality can be written as a function of the volume-mass properties of the soil.

The excess pore-air and pore-water pressures generated by a change in total stress can be computed using the pore pressure parameters associated with the air and water phases, respectively (Hasan and Fredlund 1980).

## Laboratory tests

Laboratory experiments were performed on several compacted kaolin specimens using modified Anteus oedometers and triaxial cells (Fredlund 1973).

**Equipment**

The modified Anteus oedometers were used to perform one-dimensional consolidation tests. The modified Wykeham Farrance triaxial cells allowed isotropic volume change testing conditions.

A high air-entry ceramic disk was sealed to the lower pedestal. This disk allows the flow of water but prevents the flow of free air. Therefore the measurement of the pore-air and pore-water pressures can be made independently. A low air-entry disk was placed on the top of the specimen to facilitate the control of the pore-air pressure. This allowed the translation of the air and the water pressures to positive values in order to prevent cavitation of the water below the high air-entry disk (i.e., the axis-translation technique (Hilf 1948)). The total, pore-air, and pore-water pressure conditions can then be controlled to study the volume change behavior of unsaturated soils.

Two rubber membranes separated by a slotted tin foil were placed around the specimen. This composite membrane was found to be essentially impermeable for a long period of time.

## Presentation of results

Five specimens of kaolin were compacted in accordance with the standard Proctor method. The initial volume-mass properties of each specimen are summarized in Table 1. Each experiment was performed by changing one of the stress state variable components: the total stress, $\sigma$, the pore-water pressure, $u_w$, or the pore-air pressure, $u_a$. The stress state component changes associated with each experiment are given in Table 2. In each case, the volume changes of soil structure and water phase were monitored.

Figure 1 shows a typical plot of volume changes due to an increment of total stress in Test 1. The plot exhibits a decrease in volume of soil structure, air and water phases during the transient process. A large instantaneous volume decrease occurred at the time when the load was applied.

An increase in the pore-air pressure (Test 2) caused the soil structure to expand temporarily (Fig. 2). The increase in air pressure; however, resulted in an increase in matric suction $(u_a - u_w)$, which in turn could cause a decrease in the soil structure at the end of the process.

A volume change process associated with a meta-stable structure is shown in Fig. 3. The decrease in air pressure (Test 3) reduced the matric suction $(u_a - u_w)$ and allowed more water to flow into the specimen. The intake of water reduced the normal and shear stresses between the soil particles. As a result, the soil structure underwent a decrease in volume (i.e., collapse phenomenon).

An increase in water pressure could cause an increase in volume of the soil structure (Fig. 4 for Test 4) if the soil had a stable structure. On the other hand, when the soil structure was meta-stable, an increase in water pressures caused the soil structure to decrease in volume or collapse (Fig. 5).

## Theoretical analyses

Attempts were made to best-ft the theoretical analyses of volume change with the results from laboratory experiments. This was accomplished by approximating the compressibilities of the soil structure, air, and water phases based on the laboratory results. Table 3 summarizes the approximate compressibilities for each of the five specimens.

The magnitudes of the compressibilities for each phase were obtained by dividing the amount of deformation at the end of each process by the change in the stress state variable. A sign (positive or negative) is attached to each

**Table 1.** Volume-mass relations for specimens tested.

| Test No | Diameter cm | Height cm | Total volume $cm^3$ | Water content % | Void ratio | Dry density $kN/m^3$ | Degree of saturation % |
|---|---|---|---|---|---|---|---|
| 1 | 10.006 | 11.815 | 929.09 | 34.32 | 1.0696 | 13.185 | 78.87 |
| 2 | 9.945 | 11.703 | 909.10 | 33.17 | 1.0251 | 13.298 | 80.61 |
| 3 | 10.543 | 5.867 | 503.53 | 29.62 | 1.2242 | 11.529 | 63.29 |
| 4 | 9.832 | 5.758 | 437.16 | 32.12 | 0.9310 | 13.281 | 90.25 |
| 5 | 6.350 | 2.283 | 72.29 | 31.18 | 1.1247 | 12.069 | 72.51 |

**Table 2.** Change in stress state variable components associated with each test.

| Test No. | Total stress, $\sigma$ kPa | | Pore water pressure, $u_w$ kPa | | Pore-air pressure, $u_a$ kPa | | Pressure change |
|---|---|---|---|---|---|---|---|
| | initial | final | initial | final | initial | final | kPa |
| 1 | 358.7 | 560.9 | 163.8 | 164.4 | 214.4 | 215.6 | $\Delta\sigma = +202.2$ |
| 2 | 560.9 | 559.0 | 164.4 | 163.1 | 215.6 | 421.1 | $\Delta u_a = +205.5$ |
| 3 | 475.1 | 476.8 | 42.9 | 42.2 | 397.8 | 206.5 | $\Delta u_a = -191.3$ |
| 4 | 611.4 | 610.1 | 177.3 | 379.2 | 532.0 | 530.9 | $\Delta u_w = +201.9$ |
| 5 | 606.7 | 605.2 | 216.5 | 323.6 | 413.8 | 413.8 | $\Delta u_w = +107.1$ |

**Fig. 1.** Soil structure and water phase volume changes associated with Test 1.

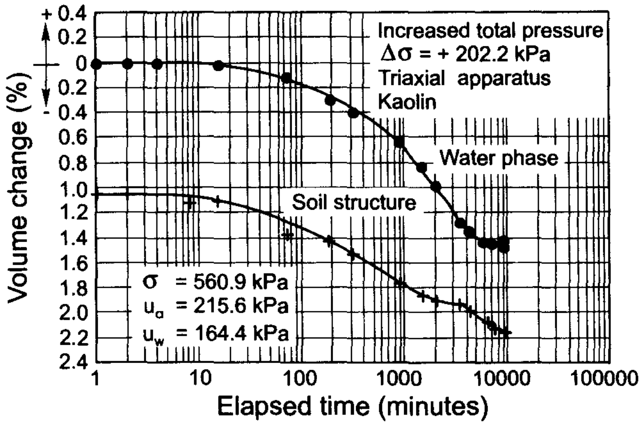



**Fig. 2.** Soil structure and water phase volume changes associated with Test 2.

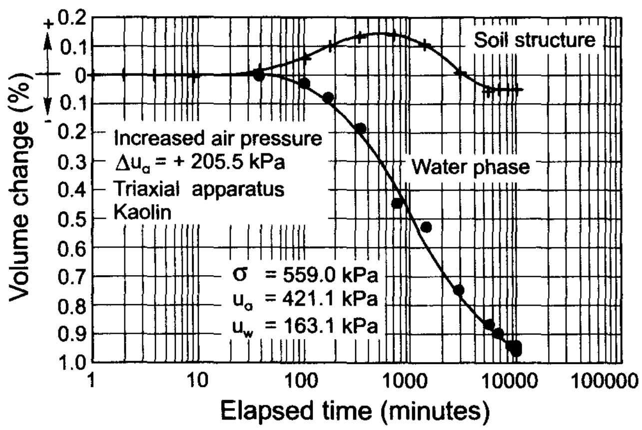



**Fig. 3.** Soil structure and water phase volume changes associated with Test 3.

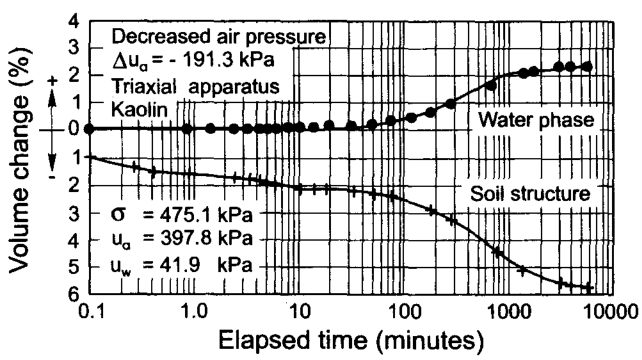



**Fig. 4.** Soil structure and water phase volume changes associated with Test 4.

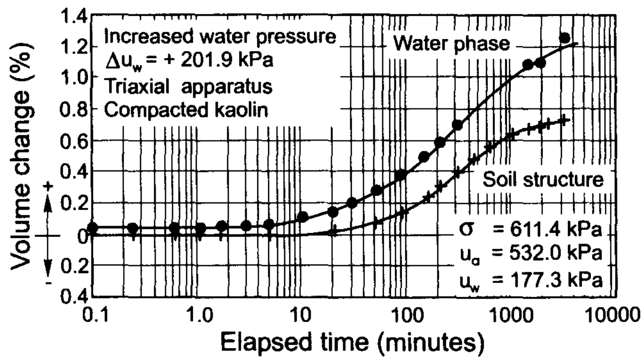
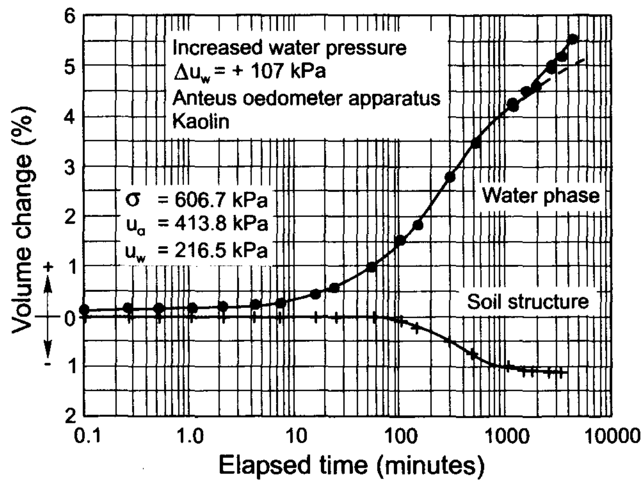



**Fig. 5.** Soil structure and water phase volume changes associated with Test 5.




of the compressibilities based on the direction (increase or decrease) of the volume change associated with each phase and the change of the stress state variables (Fredlund 1982). Table 3 indicates that the soil structure compressibilities have positive signs for a stable structured soil (Tests 1, 2, and 4) and negative signs for a meta-stable structured soil (Tests 3 and 5).

The theoretical analysis for each of the laboratory results was based upon the consideration of a one-dimensional transient flow process. The pore-water and pore-air pressures during the transient process were computed by solving simultaneously eqs. [7] and [8]. An explicit central difference technique was used for the calculations (Dakshanamurthy and Fredlund 1981). The coefficients of permeability of air and water were assumed to be constant during the transient process. Therefore the terms that account for the variation in coefficient of permeability in eqs. [7] and [8] were not used in the computations.

The volume changes associated with the water and the air phases during the transient process were computed according to eqs. [3] and [4]. The compressibilities used in the calculations were assumed to be constant throughout the process. The soil structure volume change was obtained by adding the volume changes associated with the water and air phases according to the continuity requirement in eq. [1].

Figures 6 to 10 show the theoretical analyses associated with Tests 1 to 5. The fitting was accomplished by using different combinations for the coefficients of permeability for the water and air phases. The combinations of the water and air coefficients of permeability that gave the best-fit results for each test are shown in each figure.

**Table 3.** Compressibilities for each specimen.

| Test No. | Soil structure | | Water phase | | Air-phase | |
|---|---|---|---|---|---|---|
| | $m_1^s \times 10^{-4}$ kPa$^{-1}$ | $m_2^s \times 10^{-4}$ kPa$^{-1}$ | $m_1^w \times 10^{-4}$ kPa$^{-1}$ | $m_2^w \times 10^{-4}$ kPa$^{-1}$ | $m_1^a \times 10^{-4}$ kPa$^{-1}$ | $m_2^a \times 10^{-4}$ kPa$^{-1}$ |
| 1 | 1.17 | 3.52 | 0.8 | 2.41 | 0.37 | 1.11 |
| 2 | 0.006 | 0.032 | 0.131 | 0.657 | –0.125 | –0.625 |
| 3 | –0.76 | –3.80 | 0.3 | 1.49 | –0.106 | –5.29 |
| 4 | 0.09 | 0.35 | 0.15 | 0.61 | –0.06 | –0.26 |
| 5 | –0.26 | –1.04 | 1.2 | 4.81 | –1.46 | –5.85 |

**Fig. 6.** Theoretical analysis associated with Test 1.

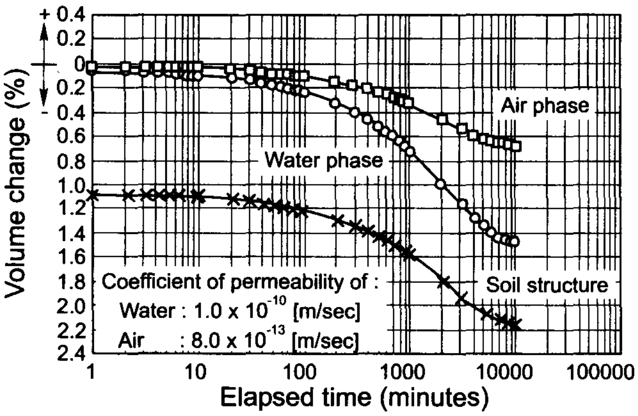



**Fig. 7.** Theoretical analysis associated with Test 2.

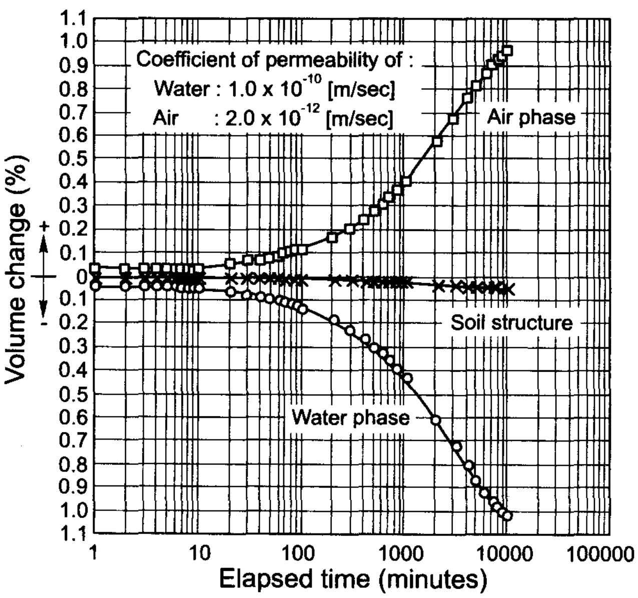



**Fig. 8.** Theoretical analysis associated with Test 3.

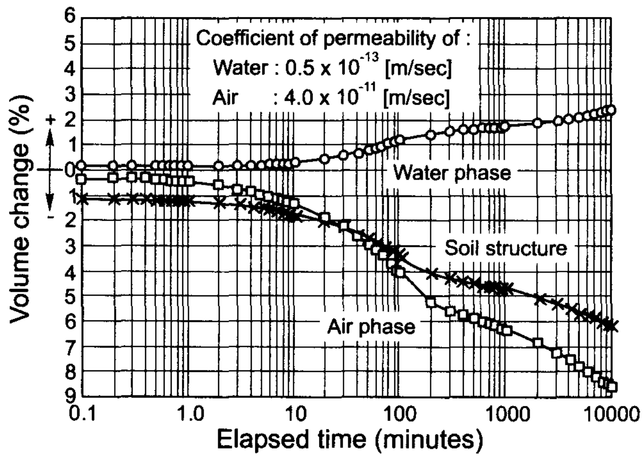
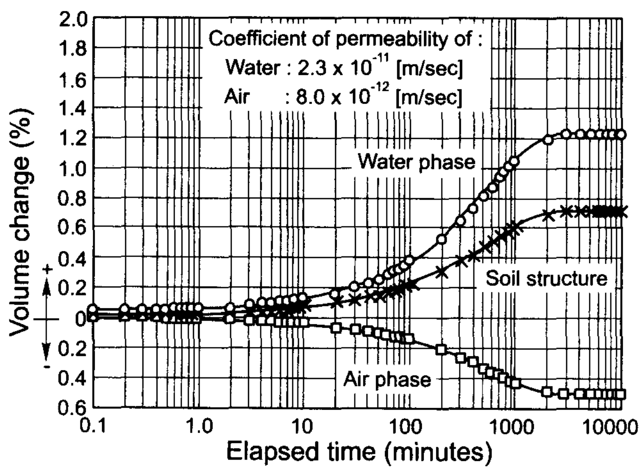



**Fig. 9.** Theoretical analysis associated with Test 4.




Comparisons between the theoretical analyses and laboratory results were made for Tests 4 and 3 (Figs. 11 and 12). The results indicate a close agreement between the theoretical analyses based on the constitutive equations and the results from the laboratory tests. Some discrepancies can be observed during the transient process. The disagreements may be due to one or more reasons. For example, the discrepancies may be due to the assumption of constant coefficients of permeability throughout the process. In spite of the difference during the process, the theoretical analyses and the laboratory results show similar trends.

## Conclusions

The comparisons between the theoretical analyses of volume change in unsaturated soils with the experimental results are shown to yield similar behavior of volume changes with respect to time. The available theory for unsaturated soils can be used to describe the volume change behavior when appropriate coefficients are used. More re-

**Fig. 10.** Theoretical analysis associated with Test 5.

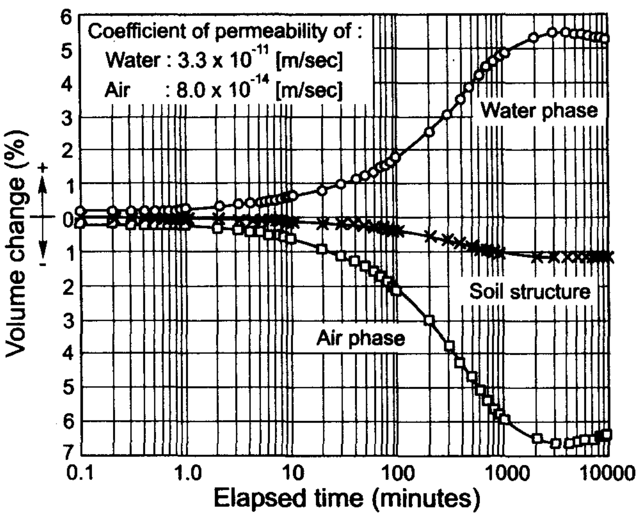



**Fig. 11.** Comparison with theoretical analyses and laboratory results for Test 4 (Fig. 4).

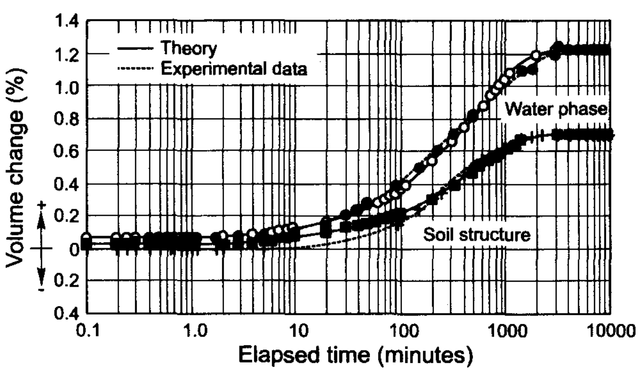
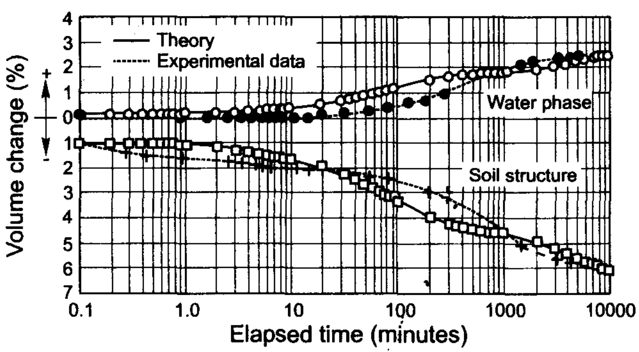



**Fig. 12.** Comparison with theoretical analyses and laboratory results for Test 3 (Fig. 3).




search should be conducted where each of the soil properties are independently measured and the time-dependent volume changes are predicted and independently monitored. The relationship between the coefficients of permeability and other soil properties also requires further study.

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## Discussion

**V. Drnevich, University of Kentucky, Lexington, KY (written discussion):**

(1) What is meant by the statement that permeability is a function of the volume-weight properties of the soil?

(2) Why were the last two terms in eq. [7] omitted when trying to fit the theory and the laboratory results?

**D.G. Fredlund and H. Rahardjo (authors' closure):**

(1) The coefficient of permeability with respect to the water phase can be written as a function of two of the commonly used volume-weight variables of the soil. It may take any one of the following forms: $k_w = f(S, e)$, $k_w = f(e, w)$, or $k_w = f(w, S)$, where $e$ = void ratio, $S$ = degree of saturation, and $w$ = water content. For an unsaturated soil, the degree of saturation or the water content is taken as the predominant variable and void ratio is assumed to be of secondary importance. Also, permeability is often written as a function of the negative pore-water pressure or matric suction. Any of the above forms are satisfactory.

(2) The laboratory tests were performed by placing specimens in contact with a high air-entry ceramic disk.

This disk has a low permeability and impedes the flow of water out of the specimen. In other words, the rate of volume change may be more an indication of the permeability of the high air-entry disks than the soil. The last two terms in eq. [7] refer to the spatial variation of permeability of the soil with time. These terns were not meaningful for the type of laboratory test performed.

Finally, we emphasize that the laboratory results presented do not provide a rigorous verification of the consolidation theory for an unsaturated soil. It is difficult to provide such a verification because of the low permeabilty of the high air-entry disk. A rigorous verification would require an independent measurement of all soil properties involved in the formulation and a measurement of the pore-air and pore-water pressure with respect to time and space. Experimentally this is extremely demanding. A lower level of verification involves the independent measurement of total and water volume changes with respect to time rather than the measurement of the pore pressures. The verification provided in this paper simply demonstrates that two independent volume changes (e.g., total volume change and water flux) occur during consolidation and that these can be modeled using two partial differential equations. Certainly there is a challenge for further verification of the consolidation theory.