A steady state model for flow in saturated–unsaturated soils

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A model is proposed describing continuous flow between saturated and unsaturated soil. The flow is assumed to be two dimensional and under steady state conditions. In the unsaturated zone, the coefficient of permeability is treated as a function of pore-water pressure head. The nonlinear differential equation governing the flow is solved using an iterative finite element scheme. The flow equation for an element is derived using the Galerkin weighed residuals method. Several example problems are solved and compared with flow net solutions. The proposed flow model is superior to traditional models, which consider flow only in the saturated zone. The results show that the zero pressure isobar is not an upper flow boundary. The finite element solution is shown to be relatively insensitive to the function used to express the relationship between the coefficient of permeability and the pore-water pressure head.

Key words: saturated–unsaturated, pore-water pressure, head, phreatic line.

Un modèle décrivant l'écoulement continu entre un sol saturé et un sol non-saturé est proposé. L'écoulement est supposé bi-dimensionnel et en régime permanent. Dans la zône non-saturée, le coefficient de perméabilité est traité comme une fonction de la pression interstitielle. L'équation différentielle non-linéaire de l'écoulement est résolue par une méthode itérative en éléments finis. L'équation de l'écoulement pour un élément est établie en utilisant la méthode des résidus compensés de Galerkin. Plusieurs problèmes types sont résolus et comparés aux solutions de réseaux d'écoulement. Le modèle proposé est supérieur aux modèles traditionnels qui ne considèrent que l'écoulement dans la zône saturée. Les résultats montrent que l'isobare de pression nulle n'est pas une frontière supérieure de l'écoulement. On montre que la solution en éléments finis est relativement insensible à la fonction utilisée pour exprimer la relation entre le coefficient de perméabilité et la pression interstitielle.

Mots clés: saturé-non – saturé, pression interstitielle, charge, ligne phréatique.

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Introduction

There is an increasing number of engineering problems involving flow through saturated–unsaturated soils. Such problems range from seepage pressure calculations in earth structures to estimates of contaminant migration in groundwater systems. The solution of these engineering problems awaits the development of rigorous flow models that will simulate these complex situations.

In seepage analysis, engineers have traditionally relied on graphical and numerical methods that consider only the saturated flow region. However, methods such as the flow net technique (Casagrande 1937) cannot adequately deal with problems involving flow through saturated–unsaturated soils.

The proposed model describes continuous flow between saturated–unsaturated soil. Flow is assumed to be two dimensional and under steady state conditions. The coefficient of permeability is treated as a function of the pore-water pressure head. The nonlinear differential equation governing the flow is solved using an iterative finite element scheme.

The theoretical formulation is implemented into a computer program called SEEP. SEEP, documented by Papagianakis (1983), can handle arbitrary degrees of anisotropy and heterogeneity and includes certain plotting capabilities for the graphical representation of the results.

Background

Some of the earliest theoretical work in the area of flow through unsaturated soils was presented by Richards in 1931. The term "capillary conduction" was used to describe the moisture movement through unsaturated soils. It was first recognized by Richards (1931) that "the essential difference between flow through a porous medium which is saturated and flow through medium which is unsaturated is that under the latter condition the pressure is described by capillary forces and the conductivity depends on the moisture content of the medium."

Casagrande (1937) proposed the flow net technique for the solution of seepage problems. According to this technique, flow problems were subdivided into "confined" and "unconfined" situations. In the case of "unconfined" flow, the upper boundary of the flow region, referred to as the line of seepage, was considered as the upper boundary of flow in the soil.

Research developments in the area of unsaturated soils have resulted in the development of relationships between the coefficient of permeability and the pore-water pressure head. Brooks and Corey (1966) derived a mathematical relationship between the coefficient of permeability and the "capillary pressure" (eq. [1]).

$$k = k_0 \qquad \text{for} \quad P_c \leq P_b$$

$$[1] \qquad k = k_0\left(\frac{P_b}{P_c}\right)^n \qquad \text{for} \quad P_c \geq P_b$$

where P_c = capillary pressure (i.e., difference between pore-air and pore-water pressure), dyn/cm^2 (1 dyn = 10 μN); P_b = bubbling pressure (i.e., minimum capillary pressure at which continuous air phase exists), dyn/cm^2; k_0 = coefficient of permeability at saturation, cm^2; k = coefficient of permeability at capillary pressure P_c, cm^2; and n = positive dimensionless constant.

Equation [1] represents two straight lines in a plot of the logarithm of the coefficient of permeability versus the logarithm of capillary pressure. Similar results were experimentally obtained by Laliberte and Corey (1967) (Fig. 1).

Taylor and Brown (1967) proposed a finite element model for seepage problems with a "free surface." This model considered the flow of water in the saturated zone only and as a result, "the principal problem is locating the free surface that has both zero flow normal to it and prescribed pressure." The proposed trial and error procedure for locating the free surface is cumbersome and often has convergence problems.

Flow models considering continuous flow between saturated–unsaturated soil were first introduced by hydrogeologists and soil scientists. Freeze (1971*a*) described a three-dimensional finite difference model for "saturated–unsaturated transient flow in nonhomogeneous and anisotropic geologic basins." The model "treats the complete subsurface region as a unified whole considering continuous flow between the saturated and unsaturated zones." The differential equation governing the flow is expressed as a function of the pore-water pressure head and is similar to the equation proposed by Richards (1931). The solution is obtained using an iterative procedure and assuming a relationship between the coefficient of permeability and the pressure head.

Subsequently, Freeze (1971*b*) compared the traditional flow models that consider the flow only in the saturated zone with the saturated–unsaturated approach. Criticizing models dealing only with the flow in the saturated region, Freeze states that "boundary conditions that are satisfied on the free surface specify that the pressure head must be atmospheric and the surface must be a streamline. Whereas the first of these conditions is true, the second is not." With respect to the continuous saturated–unsaturated flow approach, Freeze also states, "We can avoid the incorrect boundary conditions by solving the complete saturated–unsaturated boundary value problem."

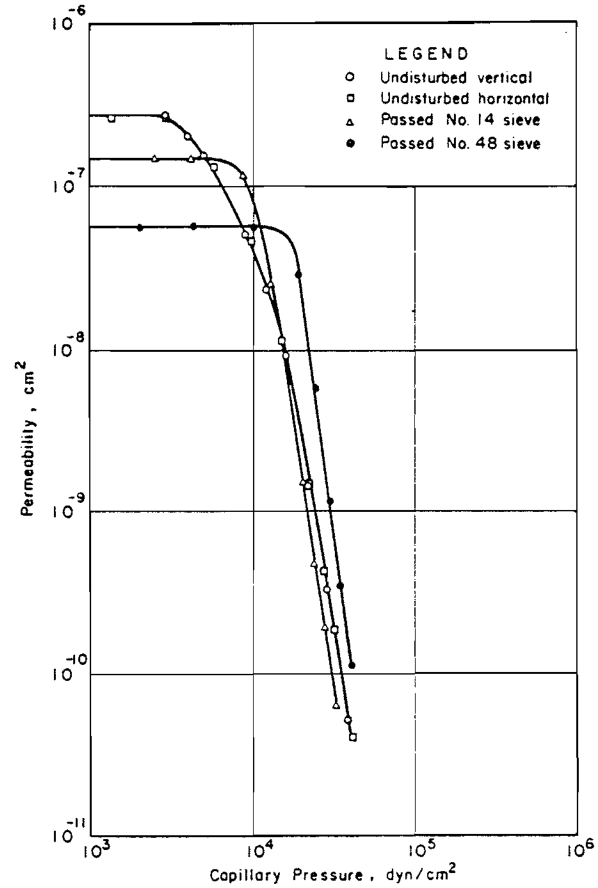


FIG. 1. Relationship between capillary pressure and coefficient of permeability; after Laliberte and Corey (1967); 1 dyn = 10 μN.

Neuman (1972) presented an iterative Galerkin finite element method for the solution of transient water flow problems in saturated–unsaturated porous media. However, the model considers the soil moisture content as the field variable, thus being rather unsuitable for engineering applications.

For the solution of steady state flow problems, engineers have traditionally relied on the flow net technique (Casagrande 1937). Flow models adopted later considered the flow of water only in the saturated zone (e.g., Taylor and Brown 1967). This "loyalty" to the concepts of "confined" and "unconfined" flow along with a lack of understanding associated with unsaturated soil behavior have discouraged the use of models considering continuous flow in saturated–unsaturated systems. Recent developments in the area of unsaturated soils (Fredlund and Morgenstern 1977; Dakshanamurthy and Fredlund 1980) offer the necessary background for using comprehensive flow models for saturated–unsaturated soils.

Theory

The differential equation governing the flow is derived assuming that flow follows Darcy's law regardless of the degree of saturation of the soil (Richards 1931). Assuming steady state flow conditions, the net flow quantity from an element of soil must be equal to zero. For the case where the direction of the maximum or minimum coefficient of permeability is parallel to either the X or Y axis, the differential equation governing the flow can be written as follows:

$$[2]\quad \frac{\partial}{\partial x}\left(k_x(u_w)\frac{\partial h}{\partial x}\right)+\frac{\partial}{\partial y}\left(k_y(u_w)\frac{\partial h}{\partial y}\right)=0$$

where h = total head (i.e., pressure head plus elevation head), m; and $k_x(u_w)$, $k_y(u_w)$ = coefficient of permeability (m/s), in the X and Y direction, respectively, depending upon the pore-water pressure head, u_w.

The nonlinear differential equation (eq. [2]) is solved using an iterative finite element scheme. In each iteration, the coefficient of permeability is assumed to be constant in an element with a value depending upon the average pore-water pressure at its nodes.

The set of flow equations for an element is derived using a Galerkin weighed residual method (Zienkiewicz 1977) (eq. [3]).

$$[3]\quad \int_A \begin{Bmatrix} \dfrac{\partial}{\partial x}\{L\} \\ \dfrac{\partial}{\partial y}\{L\} \end{Bmatrix}^T \begin{Bmatrix} k_x(u_w) & 0 \\ 0 & k_y(u_w) \end{Bmatrix} \begin{Bmatrix} \dfrac{\partial}{\partial x}\{L\} \\ \dfrac{\partial}{\partial y}\{L\} \end{Bmatrix} \mathrm{d}A\{h^n\} - \int_S \{L\}^T q\,\mathrm{d}S = 0$$

where $\{L\} = \{L_1\,L_2\,L_3\}$ = matrix of the area coordinates for an element; $\{h^n\}$ = matrix of the nodal head values for an element; A = area of an element; S = perimeter of an element; and q = flow across the sides of an element.

Assembling eq. [3] for all elements gives a set of global flow equations (eq. [4]).

$$[4]\quad \sum\{B\}^T\{k\}\{B\}A\{H^n\} - \int_S \{L\}^T q\,\mathrm{d}S = 0$$

where

$$\{B\} = \begin{Bmatrix} \dfrac{\partial}{\partial x}\{L\} \\ \dfrac{\partial}{\partial y}\{L\} \end{Bmatrix} = \frac{1}{2A}\begin{Bmatrix} y_2-y_3 & y_3-y_1 & y_1-y_2 \\ x_3-x_2 & x_1-x_3 & x_2-x_1 \end{Bmatrix}$$

$$\{k\} = \begin{Bmatrix} k_x(u_w) & 0 \\ 0 & k_y(u_w) \end{Bmatrix}$$

x_i, y_i = Cartesian coordinates of the element nodes, $\{H^n\}$ = column matrix including all the nodes of the discretized region.

The "summation" indicated in [4] is performed routinely (Desai and Abel 1972), and the surface integral is taken over the external perimeter of the flow region (Zienkiewicz 1977). When the boundary flow is zero, the second term in [4] drops to zero. When a flow value q is specified across a boundary, the surface integral in [4] distributes the surface flow into nodal flow at the corresponding boundary nodes (Segerlind 1976). The detailed finite element formulation is presented by Papagianakis (1982).

Computer implementation

The finite element formulation described above is implemented into the computer program SEEP. SEEP handles up to eight different soil materials with arbitrary anisotropy. Flow regions can also be modeled, where the direction of the major coefficient or permeability is at an angle to the horizontal. The boundary conditions, that can be specified are either head or flow values at the nodes of the discretized flow region.

SEEP calculates element coefficients of permeability, water velocities, and gradients, and computes average nodal coefficients of permeability, average nodal water velocities, and average nodal gradients.

Two different functions are programmed into SEEP for the relationship between the coefficient of permeability and the pore-water pressure head. First, a linear relationship between the logarithm of the coefficient of permeability and the negative pressure head can be used (Fig. 2). Second, a linear relationship between the logarithm of the coefficient of permeability and the logarithm of pressure head, for pressure heads lower than -1.0 m, can be used (Fig. 3). Similar relationships were experimentally obtained by Richards (1931) and Laliberte and Corey (1967), respectively.

SEEP includes a method for determining the exit point of the phreatic line and adjusting the boundary conditions along free seepage boundaries. The method requires an initial guess for the exit point of the phreatic line. In subsequent iterations, the boundary conditions along the free surface are adjusted to satisfy the condition of negative pressure head for boundary nodes higher than the current exit point of the phreatic line.

Results

The solutions of several example problems are presented to demonstrate the capabilities of the proposed model (Figs. 4–15). In all cases, a total head value of 10.0 m is specified at all nodes along the wetted upstream face of the dam cross sections. For the case of dams with a horizontal drain (Figs. 4–13), a zero

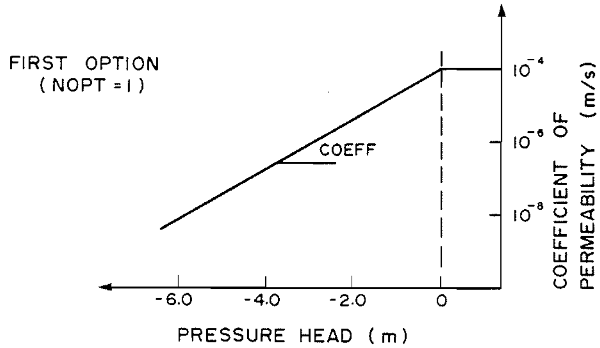


FIG. 2. Relationship between the coefficient of permeability and the pressure head (NOPT = 1).

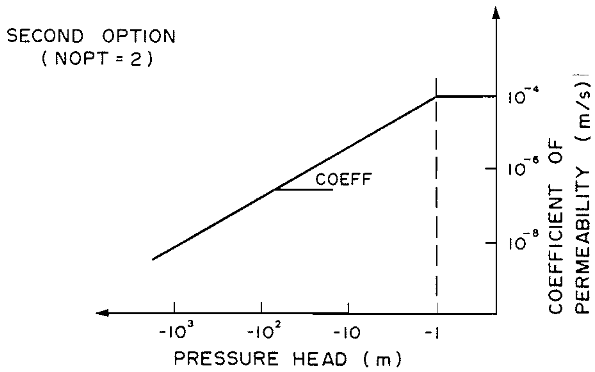


FIG. 3. Relationship between the coefficient of permeability and the pressure head (NOPT = 2).

pressure head is specified at all nodes along the drain. For the cases shown in Figs. 4–11 and 14–15, the upper flow boundary is assumed to be subjected to zero flow. For the problems shown in Figs. 12 and 13, boundary flow values of 0.1×10^{-4} m^3/s and 0.2×10^{-4} m^3/s are specified at the nodes along the upper flow boundary, respectively.

The problems shown in Figs. 14 and 15 demonstrate the ability of the computer program, SEEP, to deal with flow problems involving free boundary surfaces. In these cases, the lower boundary of the dam is assumed to be impervious and the exit point of the phreatic line is determined by the previously mentioned trial and error procedure.

For the example problems considered, a uniform grid with 195 nodal points and 336 elements was used. The number of iterations required for convergence ranges from 6 to 14, accepting a 1% tolerance (i.e., acceptable difference between corresponding nodal heads in two successive iterations). The dam geometry, the nodal head values, and the contours in Figs. 4–15 are drawn using the plotting capabilities of SEEP.

Discussion

The solutions of the example problems presented are compared with flow net solutions by observing the requirements for the phreatic line and the equipotential lines (Casagrande 1937). For all problems with zero flow specified across the upper boundary (Figs. 4–11 and 14–15), the equipotential lines coincide with the phreatic line at right angles. Furthermore, the elevation of the point of intersection of any equipotential line with the phreatic line is equal to the total head represented by this equipotential line. These observations offer evidence that the phreatic line (i.e., zero pressure isobar) approximately coincides with the line of seepage, as defined by Casagrande (1937). This latter condition will be further examined by comparing quantities of flow through various vertical sections. The problems in Figs. 12 and 13 demonstrate that when flow is applied externally, the phreatic line departs from the streamline condition.

For the problems shown in Figs. 4 and 14, the quantities of flow obtained by summing element flow quantities are compared with the flow quantities calculated from flow net solutions. The methodology for calculating flow quantities was proposed by Papagianakis (1982). The results are summarized in Tables 1 and 2, where NOPT is the type of function used and COEFF is

TABLE 1. Flow quantities (m^3/s), problem in Fig. 4

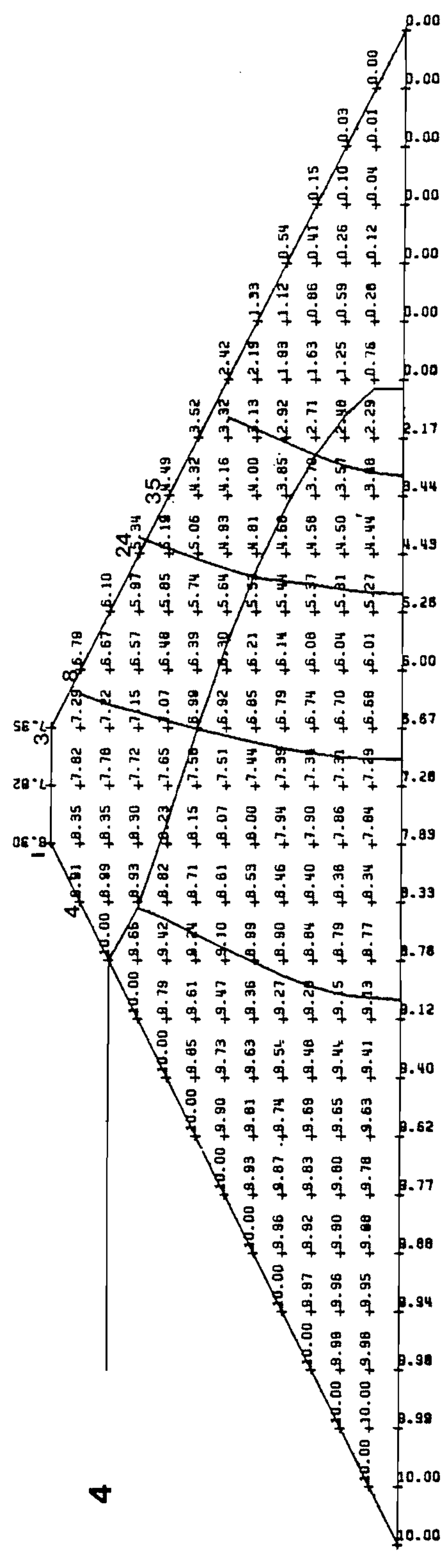
COEFF	NOPT = 1	NOPT = 2
0.2	2.645×10^{-4}	—
0.6	2.283×10^{-4}	—
1.0	2.230×10^{-4}	2.932×10^{-4}
2.0	2.175×10^{-4}	2.731×10^{-4}
3.0	—	2.636×10^{-4}
4.0	2.129×10^{-4}	2.580×10^{-4}
From flow net solution = 2.08×10^{-4} to 2.43×10^{-4}		

NOTE: NOPT = type of function (Figs. 2 or 3); COEFF = slope of function in the unsaturated zone.

TABLE 2. Flow quantities (m^3/s), problem in Fig. 14

COEFF	NOPT = 1	NOPT = 2
0.2	1.713×10^{-4}	—
0.6	1.636×10^{-4}	—
1.0	1.580×10^{-4}	1.814×10^{-4}
2.0	1.518×10^{-4}	1.798×10^{-4}
3.0	—	1.793×10^{-4}
4.0	1.482×10^{-4}	1.794×10^{-4}
From flow net solution = 1.52×10^{-4} to 1.73×10^{-4}		

NOTE: NOPT = type of function (Figs. 2 or 3); COEFF = slope of function in the unsaturated zone.



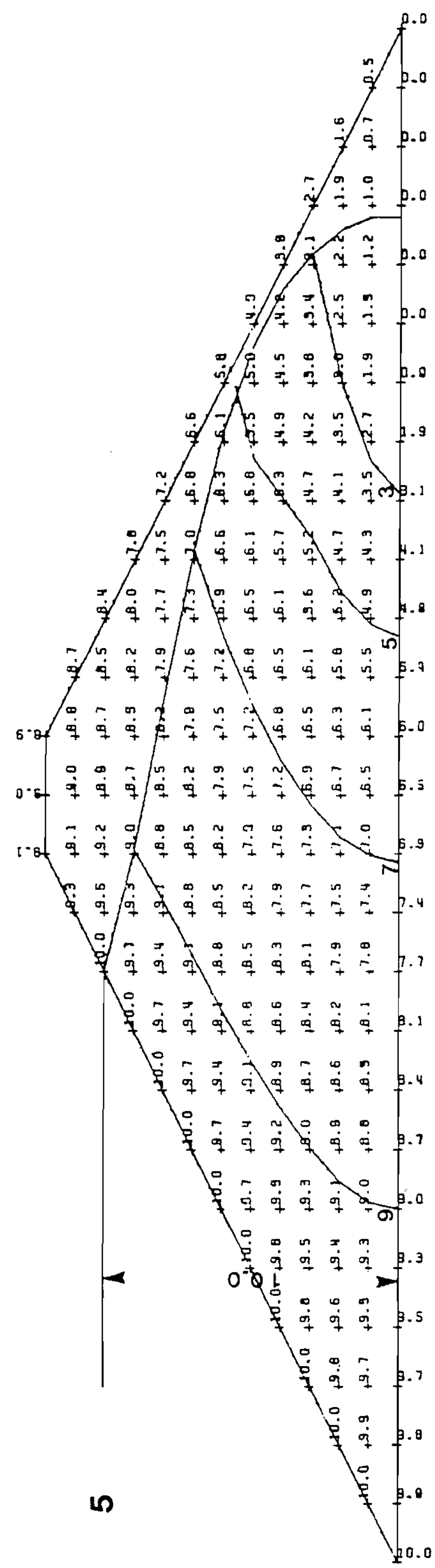


FIG. 4. Homogeneous dam with a horizontal drain.
FIG. 5. Dam with a horizontal drain; anisotropy, $k_x = 10k_y$.

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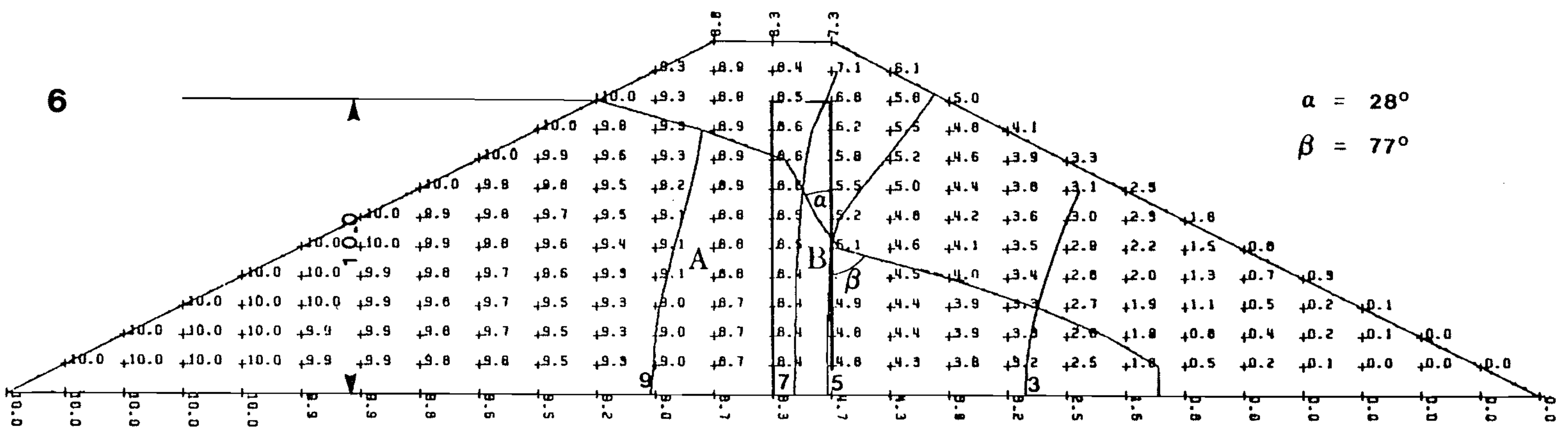

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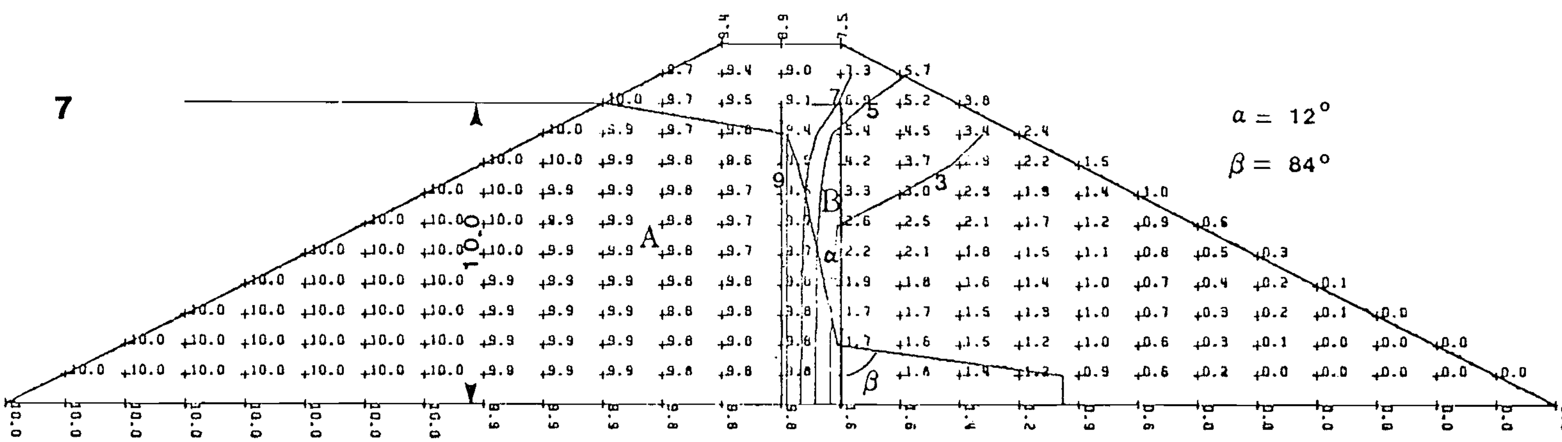
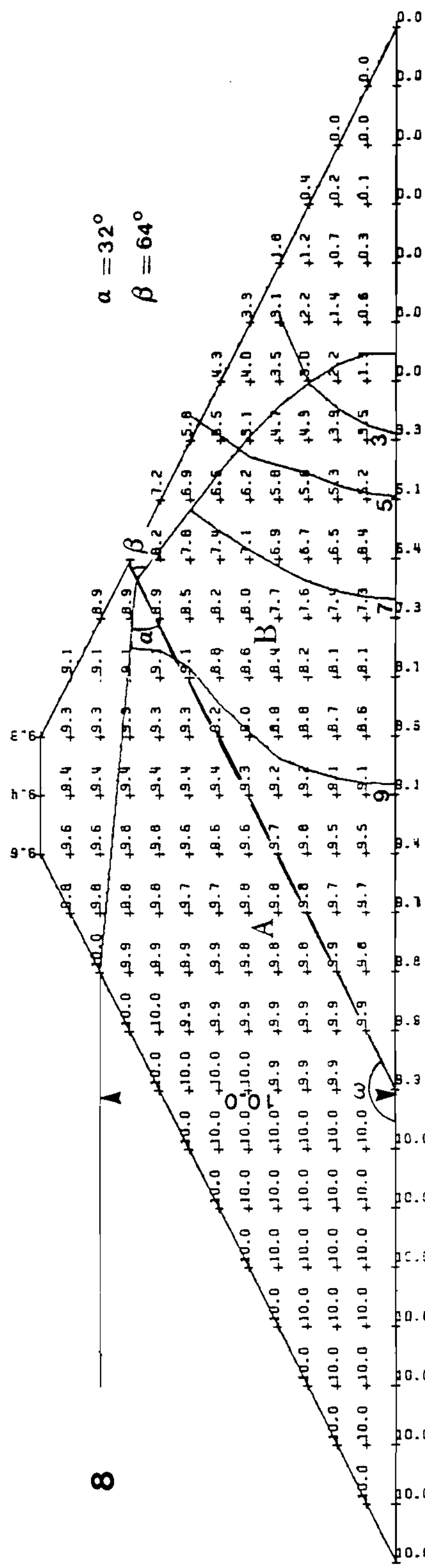


FIG. 6. Dam with a core of lower permeability; $k_A = 10k_B$.
FIG. 7. Dam with a core of lower permeability; $k_A = 10^3k_B$.



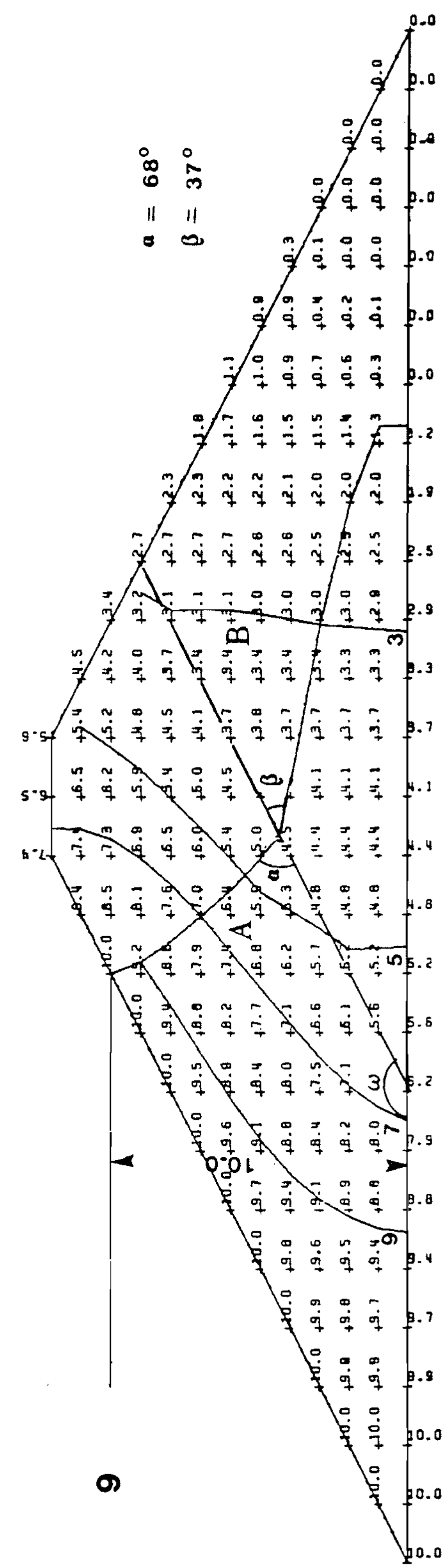
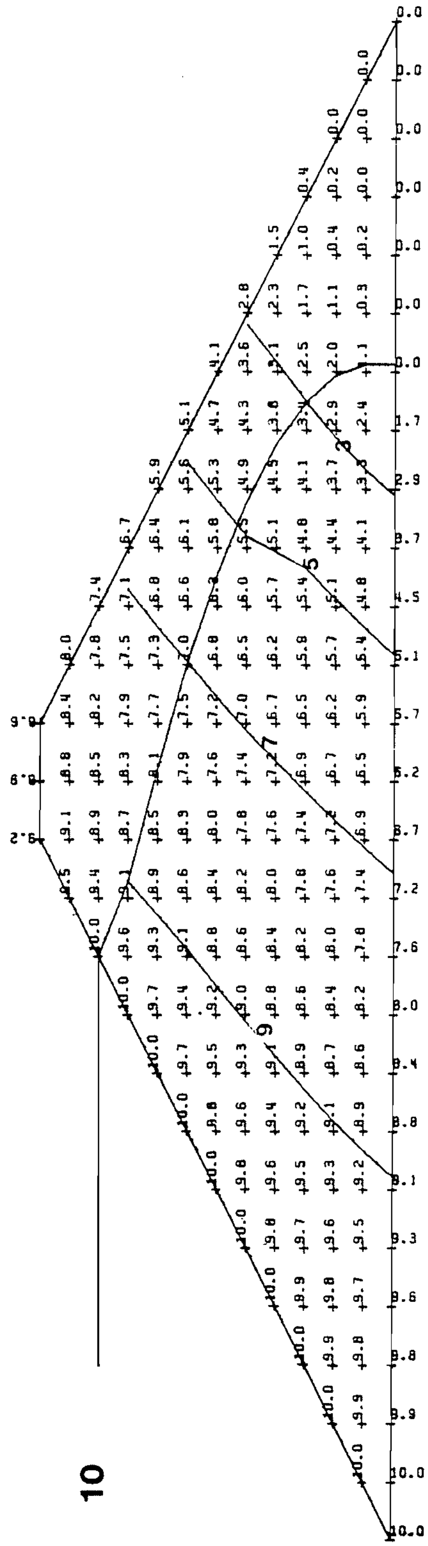


FIG. 8. Heterogeneous dam with a horizontal drain; permeabilities at saturation: $k_A = 10^{-4}$ m/s, $k_B = 10^{-5}$ m/s.
FIG. 9. Heterogeneous dam with a horizontal drain; permeabilities at saturation: $k_A = 10^{-5}$ m/s, $k_B = 10^{-4}$ m/s.



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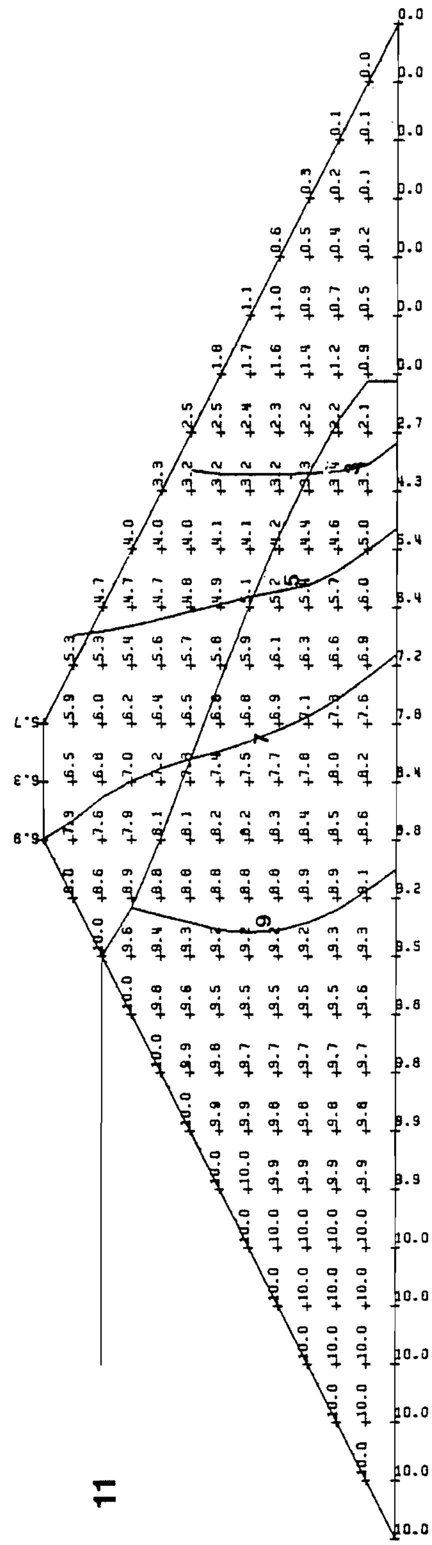

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FIG. 10. Dam with a horizontal drain; the direction of the major coefficient of permeability is at an angle 0.5 rad to the horizontal.
FIG. 11. Dam with a horizontal drain; the direction of the major coefficient of permeability is at an angle −0.5 rad to the horizontal.

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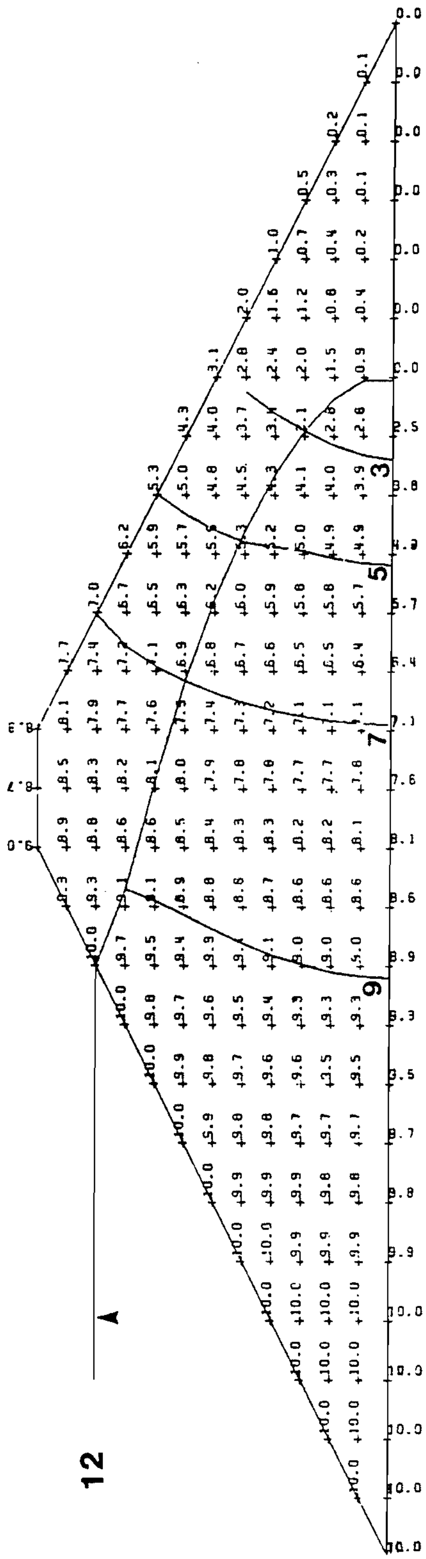

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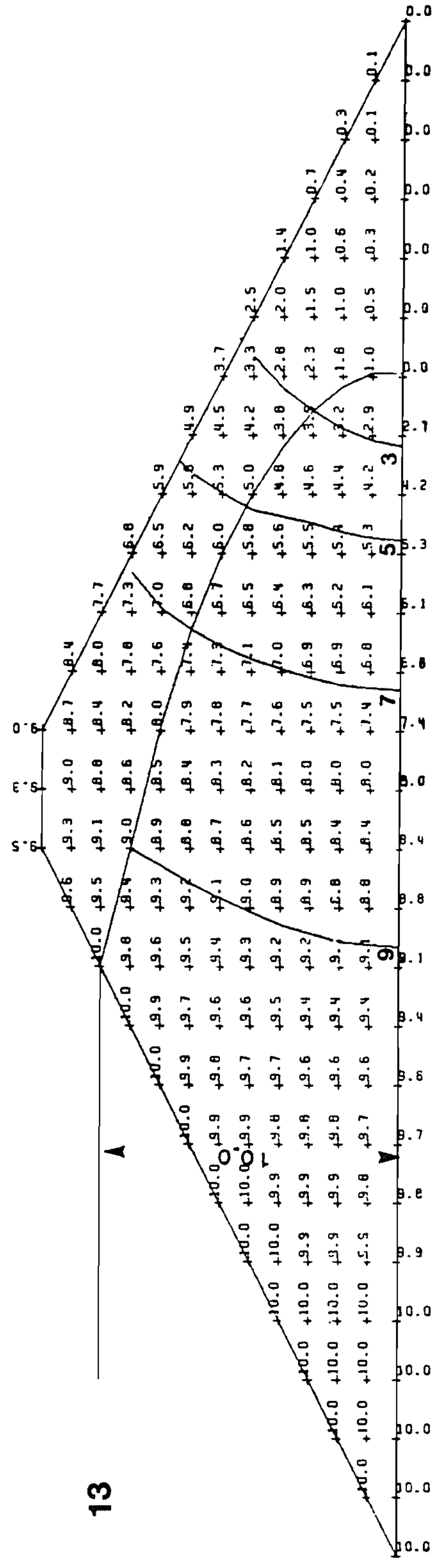
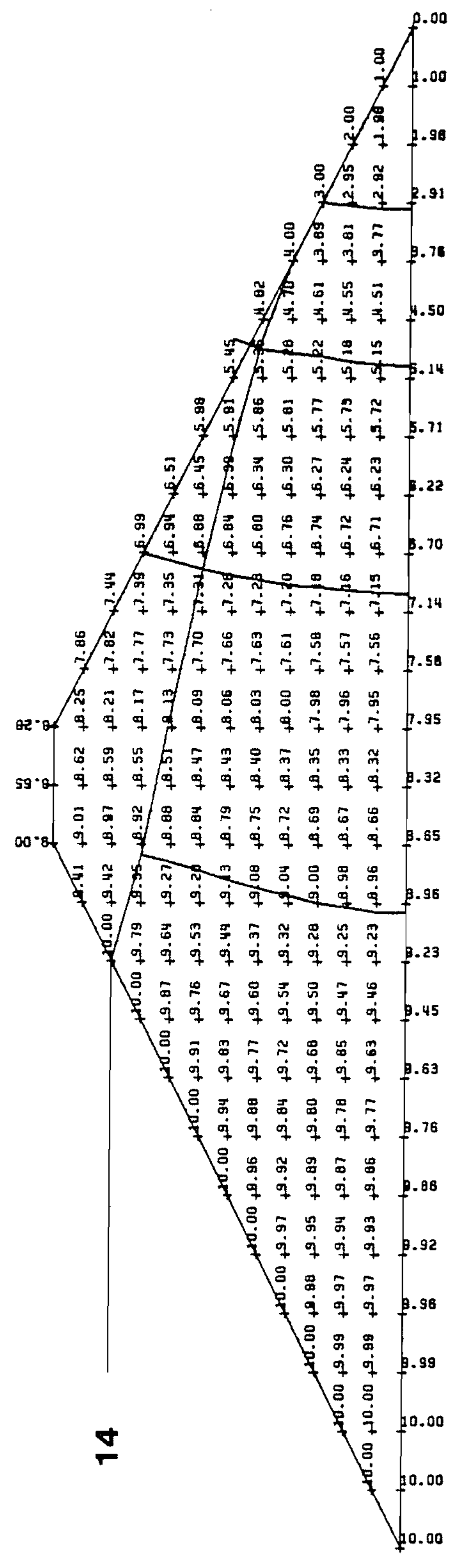


FIG. 12. Homogeneous dam under rainfall; boundary flow = 0.1×10^{-4} m/s.

FIG. 13. Homogeneous dam under rainfall; boundary flow = 0.2×10^{-4} m/s.



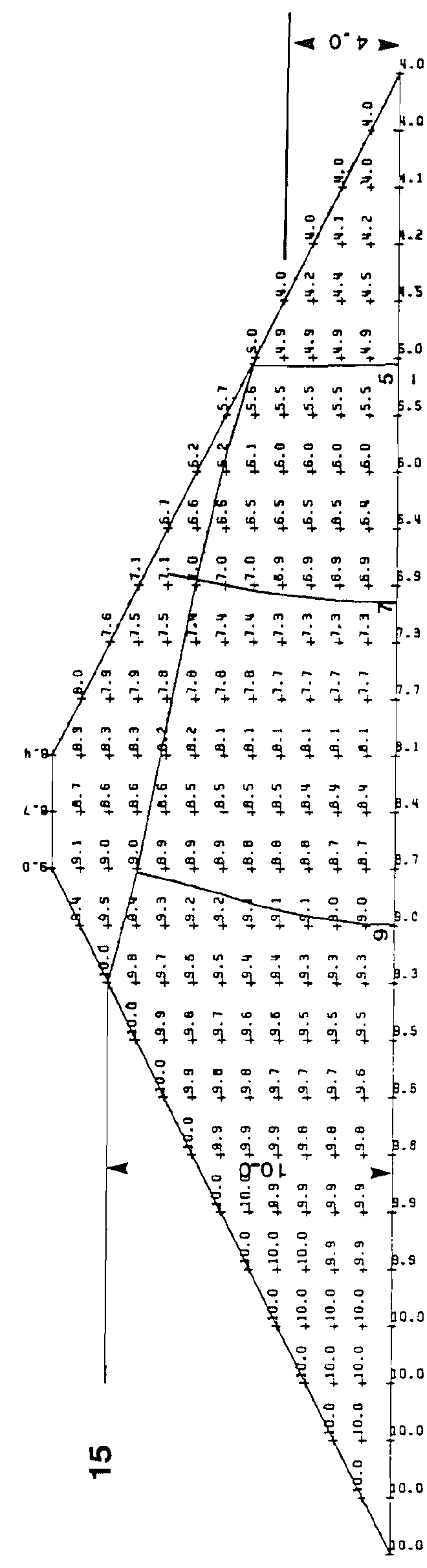


FIG. 14. Homogeneous dam with impervious lower boundary.
FIG. 15. Homogeneous dam with impervious lower boundary; 4.0 m of downstream water.

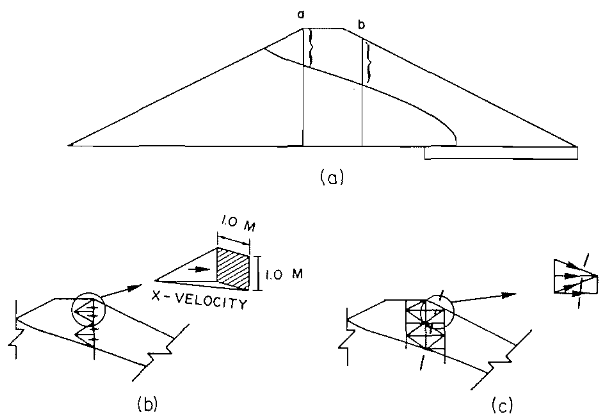


FIG. 16. Calculation of flow quantities. (*a*) Comparison of flow quantities across vertical sections; (*b*) methodology, after Papagianakis (1982); (*c*) methodology, after Lam (1984).

its slope in the unsaturated region (Figs. 2 and 3). Tables 1 and 2 show that the quantity of seepage is becoming lower for increasing COEFF values. This can be explained by the reduction of the conductivity in the unsaturated zone. It can also be seen that the quantity of seepage for NOPT = 1 and high COEFF values (e.g., 3.0 or 4.0) agrees favourably with the quantity of seepage obtained from the flow net solution. This indicates a certain dependence of the solution on the coefficient of permeability function used and suggests the need for a more detailed study on this aspect.

Figure 5 shows the effect of anisotropy on the shape of the saturated zone. For higher ratios of horizontal to vertical coefficients of permeability, the saturated zone may coincide with the downstream face of the dam.

Figures 6 and 7 illustrate that the drop of the phreatic line is becoming sharper for decreasing coefficients of permeability for the core material. However, even for three orders of magnitude difference in the coefficients of permeability, there is still some head drop taking place outside the core.

Figures 8 and 9 illustrate the deflection angles of the phreatic line at the boundary of the two materials with different coefficients of permeability. Table 3 summarizes these angles and shows that the condition $\beta = 270° - \alpha - \omega$ (Casagrande 1937) is not exactly fulfilled. The difference between the angle β and the term $270° - \alpha - \omega$ raises doubts about the streamline condition of the zero pressure isobar.

Figures 10 and 11 illustrate the distorted shape of the phreatic line and equipotential lines in flow regions where the direction of the maximum coefficient of permeability is at an angle to the horizontal.

In the cases where positive boundary flow is specified (Figs. 12 and 13), the location of the phreatic line is somewhat higher than in the case where zero flow is specified (Fig. 4).

TABLE 3. Deflection angles (deg) of the phreatic line for examining the condition $\beta = 270° - \alpha - \omega$

	α	β	$270° - \alpha - \omega$
Fig. 8	32	64	84
Fig. 9	68	37	48

NOTE: ω = 135° (Figs. 8 and 9).

Figures 14 and 15 illustrate the use of the previously mentioned procedure for revising the boundary conditions along an open flow surface. It can be seen that the presence of the downstream water causes the exit point of the phreatic line to rise. This shows that the exit point of the phreatic line in no case coincides with the free surface at the downstream water level.

The limitations of the traditional saturated-only flow models, in terms of computing effort and convergence difficulty, are well known (Taylor and Brown 1967). Furthermore, the assumption such models employ with respect to the line of seepage should be questioned. Is the zero pressure isobar (i.e., phreatic line) a streamline?

The concept of zero flow across the phreatic line is examined by comparing the quantities of flow across various vertical sections (Fig. 16*a*). Flow quantities across vertical sections are computed using two different methodologies, one proposed by Papagianakis (1982) and the other by Lam (1984). According to the first methodology, the flow quantity is calculated by summing the products of element horizontal velocities multiplied by the cross-sectional areas of these elements (Fig. 16*b*). According to the second methodology, the flow quantity is calculated by summing internodal flow quantities (Fig. 16*c*). For the example problem shown in Fig. 4, the results of the flow calculations for three vertical sections are tabulated in Tables 4 and 5. It can be seen that for both methodologies, the quantities of flow across the sections in the unsaturated zone are not equal. This implies that there is flow across the phreatic line, which as a result is not a streamline. Thus, the assumption of zero flow across the phreatic line is not only unnecessary, but it is also incorrect. This conclusion is in agreement with the work by Freeze (1971*b*). Considering that total flow quantities through various cross sections should be constant, it appears that the second methodology is more accurate.

The sensitivity of the finite element solution with respect to the function used to express the relationship between the coefficient of permeability and the pressure head was part of an extensive parametric study by Papagianakis (1982). The two problems shown in Figs. 4 and 14 were solved using various slopes of the two types of linear relationships in the unsaturated zone (Figs. 2 and 3). The solution in all cases converged in less that 20 iterations for a 1% tolerance. It was also observed that

TABLE 4. Flow quantities (10^{-4} m^3/s) across three vertical sections, methodology, after Papagianakis (1982)

	At node 3	At node 4	At node 35
Unsaturated zone	0.5585	0.3642	0.6961
Saturated zone	2.2560	2.3890	2.1728
Total	2.8145	2.7532	2.8689

TABLE 5. Flow quantities (10^{-4} m^3/s) across three vertical sections, methodology, after Lam (1984)

	At node 1	At node 8	At node 24
Unsaturated zone	0.5252	0.6557	0.5913
Saturated zone	2.2368	2.1063	2.1707
Total	2.7620	2.7620	2.7620

the number of iterations required for convergence increased for steeper functions. The calculated total head values and the location of the phreatic line were relatively unchanged under a wide range of functional relationships. The water conductivity of the unsaturated zone, however, decreased for increasing COEFF values. It is, therefore, concluded that the relative quantity of flow in the saturated and the unsaturated zone depends upon the function used. As expected, the solution obtained for a low conductivity in the unsaturated zone (i.e., high COEFF values) agrees favourably with Casagrande's solutions (1937).

Conclusions

The proposed model appears to adequately describe two-dimensional steady state flow through saturated–unsaturated soils. The computer program SEEP was proven to be relatively flexible and suitable for modelling various soil and boundary conditions. The main conclusions are as follows: (1) There is no need to retain the arbitrary categorization of flow problems as "confined" and "unconfined" situations. (2) There is continuous flow between the saturated and the unsaturated zone. (3) It is incorrect to assume that the zero pressure isobar is the upper streamline and that no flow crosses it.

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