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SATURATED-UNSATURATED TRANSIENT FINITE ELEMENT SEEPAGE MODEL FOR GEOTECHNICAL ENGINEERING

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INTRODUCTION

The analysis of seepage through porous media is of interest to many disciplines of engineering. In geotechnical engineering, seepage analysis is required in slope stability analysis, in groundwater contamination control and in the design of various hydraulic structures. Geotechnical engineers have tended to focus their attention on the saturated zone in seepage analysis.

Actually, unsaturated flow plays an important role in many engineering problems. For example, the movement of toxic leachates beneath tailings impoundment and sanitary landfills take place mainly under unsaturated conditions (Neuman, 1973). The transient analysis of seepage through a dam due to fluctuations in reservoir level is also highly influenced by the condition of the unsaturated zone (Freeze, 1971). Furthermore, in groundwater seepage problems involving infiltration and evaporation, realistic solutions can only be obtained if the unsaturated zone is properly taken into account.

The problems of water flow in saturated-unsaturated soils lead to nonlinear partial differential equations that are difficult to solve by graphical or analytical methods. With the development of high speed digital computers, numerical methods are increasingly used in solving the governing differential equation.

A two-dimensional transient finite element seepage analysis model is proposed. Two example problems are presented to demonstrate the capabilities of the model.

BACKGROUND

Some of the earliest theoretical work in the area of flow through unsaturated soils was presented by Richards in 1931.

Richards (1931) recognized that Darcy's law, which was originally proposed for saturated soils can equally be applied to unsaturated soils. The essential difference is that the coefficient of permeability is a constant for saturated soil, while it is a function of suction or moisture content for the unsaturated soil.

In 1937, Casagrande proposed a graphical method for the solution of seepage problems. The flow-net method of solution is practical when the boundary of the region under analysed is clearly defined and the soil medium is homogeneous and isotropic. However, as is often the case, when the region is non-homogeneous and highly anisotropic, a solution by means of sketching flow nets is not practical. Furthermore, Casagrande (1937) assumed that water flowed only in the saturated zone of the porous media. As a result, the line of seepage was considered to be the uppermost flow line, and there should be no water flow across the line of seepage from the saturated zone to the unsaturated zone.

In 1958, Gardner proposed a general relationship between the coefficient of permeability and the capillary pressure for unsaturated soils. In general, Gardner's equation can be written as:

$$K = \frac{K_s}{1 + a|h|^n} \qquad (1)$$

where: K_s = coefficient of permeability at saturation.
h = negative water pressure head (suction).
n = positive dimensionless constant.
a = constant depending on the units of permeability and suction.

Taylor and Brown (1967) presented a solution by finite element in seepage through earth dam with a free surface. They assumed that water flowed in the saturated zone only and the free surface was the uppermost flow line. They stated, "the principal problem is locating the position of the surface that has both zero flow normal to it and prescribed pressure". To locate the position of the free surface, Taylor and Brown (1967) proposed a special trial and error procedure. The procedure was tedious and required a considerable amount of computing time. For the case of transient state, solution by this procedure is difficult.

In 1971, Freeze developed a three-dimensional finite difference model for transient flow through saturated-unsaturated soils. The complete subsurface regime was treated as a unified whole because the flow in the unsaturated zone was integrated with the saturated flow. Consequently, instead of treating the free surface as the uppermost boundary, the ground surface was taken as the uppermost boundary. Furthermore Freeze

(1971) demonstrated that the free surface was not the uppermost flow line. It was stated to simply be a zero pressure isobar in the flow region which delineates the boundary of the saturated and unsaturated zone. Freeze (1971) also showed that the conditions of the unsaturated zone may stronly influence the position of the phreatic line, and failure to include the unsaturated zone can lead to results that are in error.

Most of the research concerning unsaturated flow have been undertaken by soil scientists. Geotechnical engineers have accepted analyses from different backgrounds without retaining a consistent framework for their formulation. Fredlund and Morgenstern (1976) provided the understanding of stress states and the relevant constitutive equations for an unsaturated soil. Furthermore, in 1981, Fredlund presented a transient flow theory through unsaturated soil with a framework most familiar to geotechnical engineers. Consequently, all the necessary physical relations for the formulation of a rigorous transient flow model through saturated-unsaturated soil systems are available.

In 1982, Papagianakis proposed a two-dimensional finite element model for flow in saturated-unsaturated soils. A computer program, SEEP, was developed for steady state flow conditions. Lam (1983) extended the research to a general steady and transient state finite element flow model called TRASEE.

THEORY AND FINITE ELEMENT FORMULATION

The proposed model describes the flow of water in a two-dimensional saturated-unsaturated soil systems. In the unsaturated zone, the air phase is assumed continuous and under atmospheric pressure. Therefore, the model considers only the flow of water. For both saturated and unsaturated soil, the water discharge velocity is assumed to be defined by Darcy's law. In the unsaturated soil, it is assumed that the coefficients of permeability of the medium depends on the negative pressure head. The formulation of the model is based on the Galerkin principle of the weighted-residual method. To integrate with time, the time-centered (Crank-Nicolson) iterative scheme is used.

Governing Differential Equation

The differential equation governing the flow is derived by equating the net flow quantity from a soil element to the time derivative of the constitutive relationship for an unsaturated soil (Fredlund and Morgenstern, 1976).

$$\frac{\partial}{\partial x}\left[k_{xx}\frac{\partial h}{\partial x} + k_{xy}\frac{\partial h}{\partial y}\right] + \frac{\partial}{\partial y}\left[k_{yx}\frac{\partial h}{\partial x} + k_{yy}\frac{\partial h}{\partial y}\right] = \rho_w g m_2^w \frac{\partial h}{\partial t} \quad (2)$$

where: k_{xx}, k_{xy}, k_{yx}, k_{yy} = permeability tensor of an anisotropic soil
h = total head
m_2^w = slope of the $(u_a - u_w)$ versus Θ_w when $d(\sigma - u_a) = 0$
u_a = pore-air pressure
u_w = pore-water pressure
Θ_w = net inflow or outflow of water for the element
σ = total stress
ρ_w = density of water
g = acceleration of gravity

For steady state conditions, the right-hand side of equation (2) goes to zero. Equation 2 is nonlinear because the values of the coefficient of permeability are not constant but depend upon the unknown pressure head.

Flow Equation for an Element

Using the Galerkin principle of weight-residuals, the flow equation for an element is derived as follows:

$$\int_A \{B\}^T [K] \{B\} dA \{h^n\} + \int_A \{L\}^T \lambda \{L\} \frac{\partial \{h^n\}}{\partial t} dA - \int_S \{L\}^T q ds = 0 \qquad (3)$$

where: $\{B\} = \frac{1}{2A} \begin{Bmatrix} y_2 - y_3, & y_3 - y_1, & y_1 - y_2 \\ x_3 - x_2, & x_1 - x_3, & x_2 - x_1 \end{Bmatrix}$, with x_i, y_i the Cartesian coordinates of the nodes of the element.

$[K] = \begin{bmatrix} k_{xx} & k_{xy} \\ k_{yx} & k_{yy} \end{bmatrix}$, with k_{xx}, k_{yy}, k_{xy}, k_{yx} the components of permeability tensor, K

$\{h^n\} = \begin{bmatrix} h_1 \\ h_2 \\ h_3 \end{bmatrix}$, with h_i the total head at the nodes of the element

$\{L\}^T = \{L_1, L_2, L_3\}$, with L_i the area coordinates

$\lambda = \rho_w g m_2^w$
q = flow across the perimeter of the element
A = area of the element
s = perimeter of the element
t = time

Integration Over Time

For transient flow, equation 3 is integrated in time using the time centered, Crank-Nicolson finite difference scheme (Zienkiewicz, 1977). The relationship between nodal heads in two successive time steps of an element can be expressed as follows:

$$\left[\frac{2}{\Delta t}[C] + [D]^{t+\Delta t/2}\right]\{h^n\}_{t+1} = \left[\frac{2}{\Delta t}[C] - [D]^{t+\Delta t/2}\right]\{h^n\}_t + 2\{F\}^{t+\Delta t/2} \qquad (4)$$

where: $[D]^{t+\Delta t/2} = \{B\}^T [K]^{t+\Delta t/2} \{B\}$ A, with D being the stiffness matrix evaluated at $t+\Delta t/2$

$$[C] = \frac{\lambda A}{12}\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$, with C being the capacitance matrix

$\{F\}^{t+\Delta t/2}$ = load vector reflecting the boundary conditions evaluated at $t+\Delta t/2$

$\{h^n\}_t$ = known nodal total heads at time, t

$\{h^n\}_{t+1}$ = unknown nodal total heads at time, t+1

Global Head - Flow Equation

Equation 4 can be written in a simpler expression as:

$$[S]\,[h^n]_{t+1} = [Y] \qquad (5)$$

After matrix [S] and [Y] in equation 5 for an element are formed, the next step is to construct the algebraic equations for the whole system from the equations for the individual element. Nodal compatibility is used as the basis for this process (Desai and Abel, 1972). This simple requirement states that all elements adjacent to a particular node must have the same total head at that node. The set of linear simultaneous equations given by equation 5 are solved with respect to nodal total head using Gaussion elimination technique.

COMPUTER IMPLEMENTATION

The formulated finite element solutions are computer implemented in the program TRASEE. The computer facility used is the DIGITAL VAX 11/780 system at the College of Engineering, University of Saskatchewan, Saskatoon.

TRASEE is a two-dimensional finite element computer program that can model both steady state and transient flow conditions. TRASEE can accommodate complex geometries with arbitrary degrees of heterogeneity and anisotropy for up to 12 different soils. The present version of TRASEE has the capability of modelling a discretized soil system of up to 1300 elements and 800 nodes. Anisotropic conditions can also be handled where the direction of the major coefficient of permeability is at any specified angle to the x-axis.

Boundary conditions of either total head or flow can be specified at certain nodes. Flow boundary conditions such as infiltration and evapotranspiration and pumping can be simulated by designating a positive or negative flow at certain nodes. Changing boundary conditions during the transient process can be handled as a prescribed step function. A special procedure is also included for revising the boundary conditions along seepage faces. Furthermore, the program also has the capability to calculate the flux through any sections of the soil system.

For every time step, TRASEE calculates the total head at each node, then calculates element nodal velocities and gradients in the x and y directions. Finally, the flux quantity through different sections are calculated. Auxiliary plotting programs have been written to facilitate in the interpretation and presentation of the computed results.

RESULTS

Two example problems are presented. Example 1 is seepage through a homogeneous dam with a horizontal drain. At time equal to zero, the reservoir level is raised from 4 meters to 10 meters instantaneously. Figure 1a illustrates the rising of the phreatic lines with time. Figures 1b and 1c present the velocity field and the total nodal head values 32 minutes after the raising of the reservoir level. Figures 1d and 1e present the solution at final steady state condition.

Example 2 involves groundwater seepage under a tailings impoundment. Initially the pond is empty, and the soil underneath the pond is in equilibrium with the water table located at 5 meters below ground surface. At a time equal to zero, the pond is assumed to be filled with tailings which set up a water pressure head of 1.0 meter. Figure 2a illustrates the position of the groundwater table due to the influence of the tailings pond with time. Figures 2b to 2d present the velocity field at different times during the transient flow process.

DISCUSSION AND CONCLUSION

The fact that the phreatic line is not a flow line is confirmed by the velocity fields presented in Figures 1b and 1d. In these figures, the average nodal velocity vectors indicate that there is a considerable amount of water flow across the phreatic line. As is shown in Figures 1c and 1e, the equipotential lines extend all the way through the unsaturated zone. This means that there is potential gradient in the unsaturated zone, and there can be water flow in the unsaturated zone. This phenomena agree with the solution by Freeze (1971).

The results have also been compared with Casagrande's classic solutions based on the flow net technique. The solution of the final steady state condition (Figure 1e) compares satisfactorily with the flow net solution. The small differences are due to the fact that the model takes into account water flow in the unsaturated zone, while it is not considered using the flow net technique.

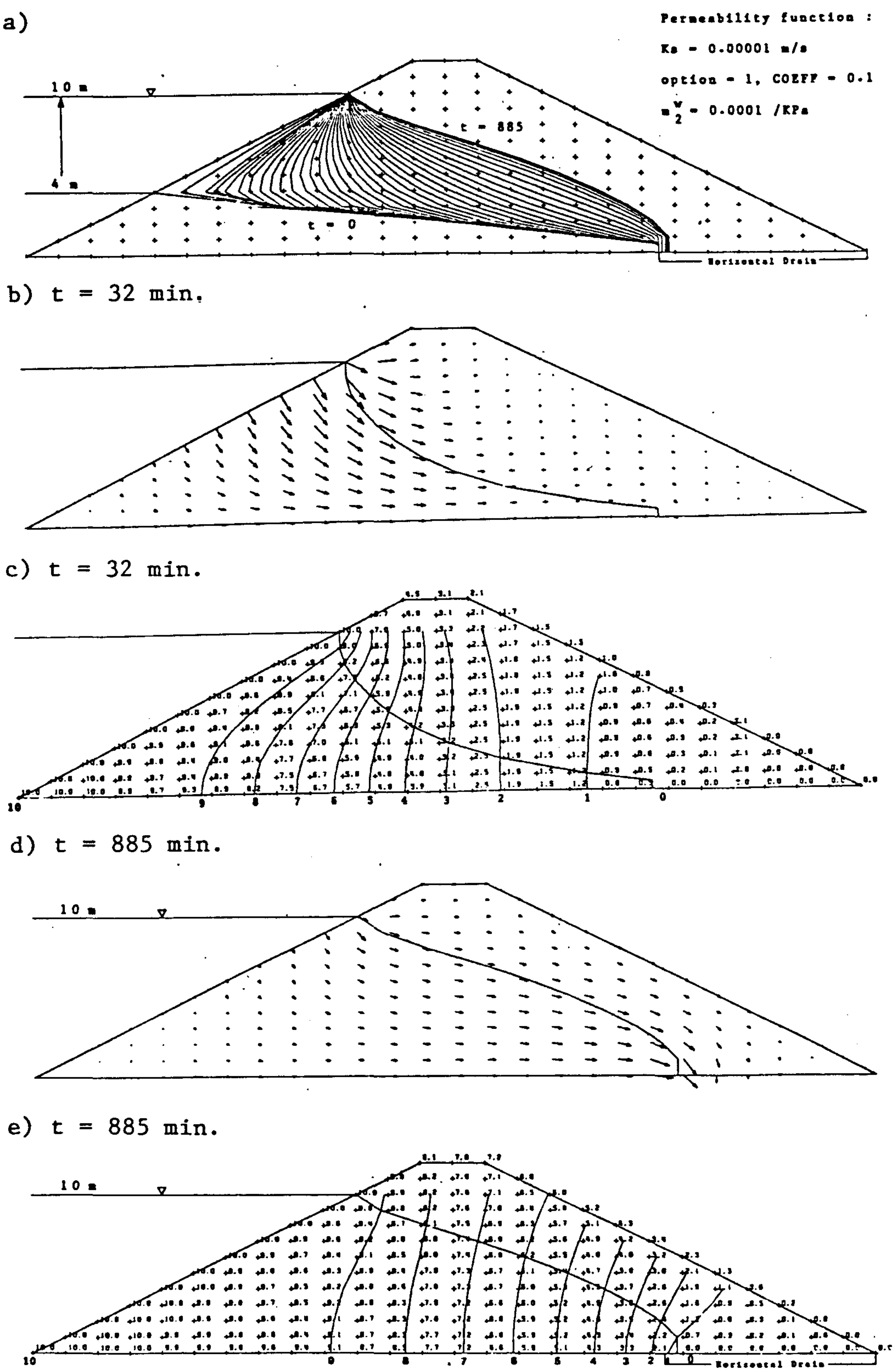
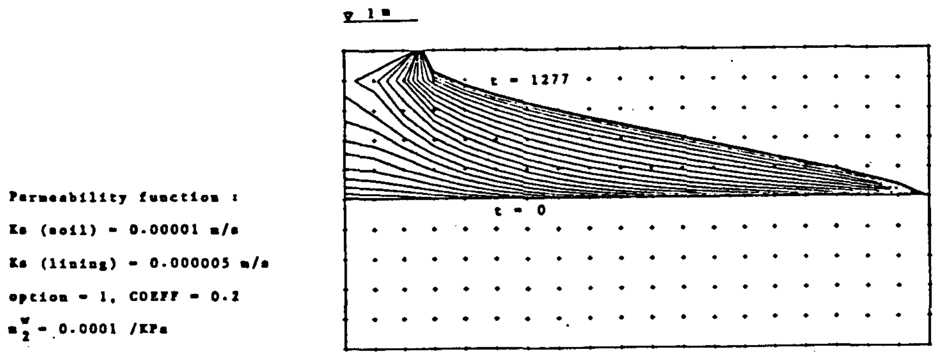
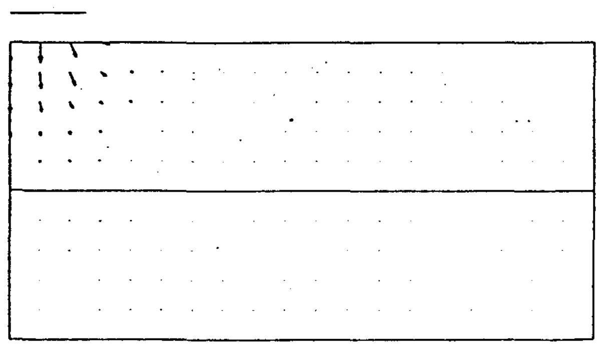


FIGURE 1 SEEPAGE THROUGH HOMOGENEOUS DAM WITH HORIZONTAL DRAIN
a) RISING OF PHREATIC LINES WITH TIME
b) to e) VELOCITY FIELD AND TOTAL HEAD AT DIFFERENT TIME

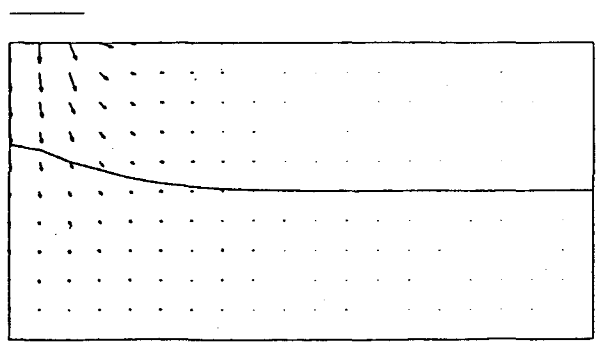
a)



b) t = 6.5 min.



c) t = 22 min.



d) t = 1277 min.

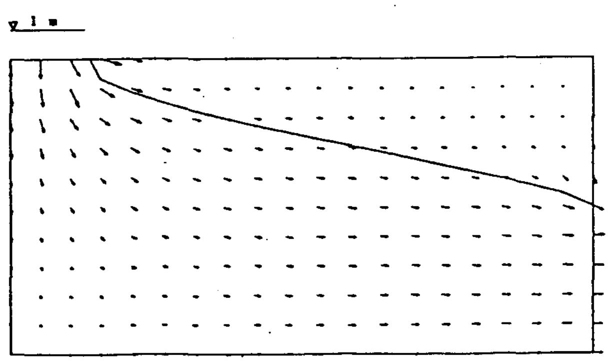


FIGURE 2 GROUNDWATER SEEPAGE UNDER TAILINGS IMPOUNDMENT
a) RISING OF WATER TABLE WITH TIME
b) to d) VELOCITY FIELD SHOWING THE THREE SEEPAGE STAGES

Example 2 is a class of problem to which the conventional "saturated only" methods of analysis cannot be applied. Generally, seepage under an impoundment is considered as taking place in three stages. These three stages can be illustrated by Figures 2b to 2d. Stage 1 (Figure 2b) is the advance of water from the impoundment through the unsaturated zone to the saturated zone which is delineated by the water table. Stage 2 (Figure 2c) is when the water contacts the water table and starts building up a groundwater mound toward the impoundment. Stage 3 (Figure 2d) occurs if the groundwater mound comes in contact with the impoundment. At this stage, the soil underneath the impoundment is saturated, and water flows from the impoundment directly into the saturated zone.

The transient solution of example 2 compares satisfactory with the transient process presented by McWhorter and Nelson (1979). Similar types of analysis can be performed for tailings impoundment design and for the prediction of contaminant migration into the groundwater regime.

The traditional "saturated-only" approach was found to be a special case to the saturated-unsaturated model. The saturated-unsaturated model provides a more realistic representation of the actual situations, and applies to a much wider range of engineering problems.

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