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Quantitative Comparison of Limit Equilibrium Methods of Slices

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SYNOPSIS

A comprehensive study is presented to compare the factors of safety calculated by the Ordinary, Bishop's Simplified, Janbu's Simplified, Janbu's Generalized, Spencer and Morgenstern-Price methods. Results are presented in terms of stability coefficients m and n.

INTRODUCTION

Numerous methods of slices have been developed to analyze slope stability problems. Although these methods are based on the principles of static equilibrium, the results differ due to different statics and assumptions used in the analysis. Bishop (1955) showed that discrepancies in the factors of safety computed by the Ordinary and the Bishop's Simplified methods were significant, in some cases as much as 60%. Janbu (1980) compared a number of analytical procedures and concluded that their differences are small. Duncan and Wright (1980) showed that the factor of safety computed by a given method depends on the equilibrium condition satisfied and the side force assumptions involved. The quantitative relationship between the results from various methods has, however, not been defined.

This paper presents a quantitative comparison of the factors of safety computed using the common methods of slices. The methods of slices are examined in relation to the General Limit Equilibrium method (GLE) (Fredlund et al, 1981).

THEORETICAL ASPECTS

The methods of slices utilize the principles of static equilibrium. The number of available equations is exceeded by the number of unknown variables. Therefore, a stability analysis is statically indeterminate. Extra principles of mechanics or an assumption must be used to render the problem determinate. The latter approach has been commonly applied because of its simplicity.

The General Limit Equilibrium method of slices (GLE) provides a common basis for a comparison of the different methods and is used in this study. Figure 1 illustrates the forces acting on a slice on a general slip surface. Definitions of the variables are as follows: W = total weight of a slice of width 'b' and height 'h'; P = total normal force on the centre of the base of the slice; S_m = mobilized shearing resistance at the base of the slice; E_L, E_R = total interslice normal force on the left and right sides of the slices, respectively; X_L, X_R = interslice shear force on the left and right sides of the slice, respectively; A_L, A_R = resultant external hydrostatic forces at the left and right extremes of the slip surface, respectively; R, f, x, a = moment arms associated with the forces (Figure 1).

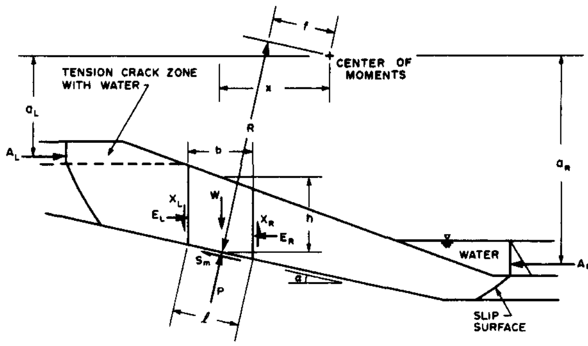

Fig. 1 Forces Applied to a Slice on a Composite Slip Surface

The mobilized shearing resistance, S_m, is computed based on the Mohr-Coulomb failure criterion for the soil.

$$S_m = \frac{1}{F} [c'\ell + (P - u\ell) \tan \phi'] \qquad (1)$$

where F = factor of safety
ℓ = length of slip surface
u = pore-water pressure, and
c', ϕ' = effective shear strength parameters.

The normal force, P, is computed by summing vertical forces for an individual slice.

$$P = \frac{W - (X_R - X_L) - \frac{c'\ell \sin\alpha}{F} + \frac{u\ell \sin\alpha \tan \phi'}{F}}{m_\alpha} \quad (2)$$

where α = inclination of the base of the slice, and $m_\alpha = \cos\alpha + \sin\alpha \tan\phi'/F$.

The factor of safety with respect to moment equilibrium, F_m, is obtained by summing moments for all the slices about an arbitrary center of moments such as the center of rotation for the circular portion of a composite slip surface.

$$\Sigma Wx - \Sigma S_m R - \Sigma Pf \pm Aa = 0 \quad (3)$$

Substituting the expressions for S_m and P from equations (1) and (2) into equation (3) gives,

$$F_m = \frac{\Sigma [c'\ell R \cos\alpha + (W - (X_R - X_L) - u\ell \cos\alpha) R \tan\phi']/m_\alpha}{\Sigma Wx - \Sigma Pf \pm Aa} \quad (4)$$

The factor of safety with respect to force equilibrium, F_f, is derived by summing horizontal forces for all the slices.

$$\Sigma P \sin\alpha - \Sigma S_m \cos\alpha \pm A = 0 \quad (5)$$

Substituting the expressions for S_m and P from equations (1) and (2) into equation (5) yields,

$$F_f = \frac{\Sigma [c'\ell + (\frac{W}{\cos\alpha} - \frac{X_R - X_L}{\cos\alpha} - u\ell) \tan\phi']/m_\alpha}{\Sigma [W - (X_R - X_L)] \tan\alpha \pm A} \quad (6)$$

The summation of horizontal forces on an individual slice is used to evaluate the interslice normal force, E.

$$(E_R - E_L) = [W - (X_R - X_L)] \tan\alpha - \frac{S_m}{\cos\alpha} \quad (7)$$

The interslice shear force, X, can be related to the interslice normal force, E, by a mathematical function (Morgenstern and Price, 1965).

$$X/E = \lambda f(x) \quad (8)$$

where $f(x)$ = interslice force function, and
λ = a scaling constant representing the percentage of the function used.

Other forms of interslice force functions can also be used. For example, Janbu's Generalized method uses the summation of moments for each slice to derive a relationship between the interslice forces.

The methods of slices considered to be commonly used are: the Ordinary, the Bishop's Simplified, the Janbu's Simplified, the Janbu's Generalized Procedures of Slices, the Spencer, the Morgenstern-Price methods and the force equilibrium methods such as Lowe-Karafiath and the Corps of Engineers methods. These methods can be compared in terms of the statics used in their derivation and can be shown to have the same normal force equation (2), moment factor of safety equation (4) and/or force factor of safety equation (6) (Fredlund and Krahn, 1977).

The difference between these methods of slices lies in the statics used and the assumptions involved regarding the interslice forces (Fredlund and Krahn, 1977). Table 1 summarizes the comparison between these methods. It is anticipated that these methods would give different results when the interslice forces become relatively significant. This frequently occurs when analyzing deep-seated slip surfaces. Moreover, methods which use force equilibrium to solve for the factor of safety may give substantially different results than those satisfying on moment equilibrium (Spencer, 1967). The Ordinary method is an exception since it completely ignores the interslice forces (Fredlund and Krahn, 1977). In this method, the normal force, P, is determined by summing forces normal to the base of each slice.

$$P = W \cos\alpha \quad (9)$$

Substituting the expression for P from equation (9) into equation (3) gives the following factor of safety equation.

$$F_m = \frac{\Sigma [c'\ell R + (W \cos\alpha - u\ell) R \tan\phi']}{\Sigma Wx - \Sigma Pf \pm Aa} \quad (10)$$

Special Cases

Special Case No. 1 ($c' = 0$)

For cohesionless soil slopes, the critical slip surface is planar and parallel to the slope surface. The inclination of the base of slice, α, is constant and equal to the slope angle, β (Haefeli, 1948). The interslice shear forces, X, vanish from the equations because these forces are conjugate. For simple slopes without external forces, the moment factor of safety equation (4) becomes,

$$F_m = \frac{1}{\Sigma Wx - \Sigma Pf} \Sigma \left[\frac{(W - u\ell \cos\alpha) R \tan\phi'}{\cos\alpha + \frac{\sin\alpha \tan\phi'}{F}} \right] \quad (11)$$

The radius, R, is constant since the centre of moments lies at infinity. Using the pore-pressure ratio, r_u, and substituting $u\ell \cos\alpha = r_u \gamma hb$ or $r_u W$ and $x = R \sin\alpha$, equation (11) becomes,

TABLE I

COMPARISON OF COMMONLY USED METHODS OF SLICES

Method	Factor of Safety Based On		Interslice Force Assumption
	Moment Equilibrium	Force Equilibrium	
Ordinary	x		$X = 0, E = 0$
Simplified Bishop	x		$X = 0, E \geq 0$
Janbu's Simplified		x	$X = 0, E \geq 0$
Janbu's Generalized		x	$X_R = E_R \tan \alpha_t - (E_R - E_L) t_R/b$ *
Spencer	x	x	$X/E = \tan \theta$ +
Morgenstern-Price	x	x	$X/E = \lambda f(x)$
Lowe and Karafiath		x	X/E = Average slope of ground and slip surface
Corps of Engineers		x	X/E = Average surface slope

* α_t = angle between the line of thrust on the right side of a slice and the horizontal.
t_R = vertical distance from the base of the slice to the line of thrust on the right side of the slice.

\+ θ = angle of the resultant interslice force from the horizontal.

$$F_m = \frac{\tan \phi'}{\tan \beta}(1 - r_u \sec^2 \beta) \qquad (12)$$

The force factor of safety equation (6) is,

$$F_f = \frac{1}{\Sigma W \tan \alpha} \Sigma \left[\frac{\frac{(W - u\ell \cos \alpha) \tan \phi'}{\cos \alpha}}{\cos^2 \alpha + \frac{\sin \alpha \tan \phi'}{F}} \right] \qquad (13)$$

Substituting $u\ell \cos \alpha = r_u W$ into equation (13) and simplifying gives,

$$F_f = \frac{\tan \phi'}{\tan \beta}(1 - r_u \sec^2 \beta) \qquad (14)$$

Similarly, the factor of safety equation (10) for the Ordinary Method becomes,

$$F_m = \frac{\Sigma[(W \cos \alpha - u\ell) R \tan \phi']}{\Sigma Wx} \qquad (15)$$

Simplifying, equation (15) reverts to an expression identical to equation (12).

The anlaysis becomes the semi-infinite slope case and all methods of slices give virtually identical results.

Special Case No. 2 ($\phi = 0$)

For cohesive soil slopes (i.e., $\phi = 0$) and in particular, relatively flat slopes, the critical slip surface is a deep-seated circle. The 'Pf' terms vanish from the moment equation since the normal forces pass through the center of moments. The moment factor of safety equations (4) and (10) for simple slopes without external forces become,

$$F_m = \frac{\Sigma c\ell R}{\Sigma Wx} \qquad (16)$$

where: $c = S_u$ = undrained shear strength.

The force factor of safety equation (6) for the zero-ϕ case is as follows.

$$F_f = \frac{\Sigma[c\ell/\cos \alpha]}{\Sigma[W - (X_R - X_L)] \tan \alpha} \qquad (17)$$

Force equilibrium methods give factors of safety dependent on the interslice force assumptions. All methods of slices that use moment equilibrium (or moment and force equilibrium) to solve for the factor of safety can be expected to give virtually identical results for this special case. Figure 2 gives a comparison of the results calculated by equations (16) and (17) for this special case. The moment equilibrium methods of slices are similar to the Friction Circle method proposed by Taylor (1937).

QUANTITATIVE COMPARATIVE STUDIES

The SLOPE-II computer program (Fredlund, 1981) was used for the comparative study because it can solve most of the commonly used methods.

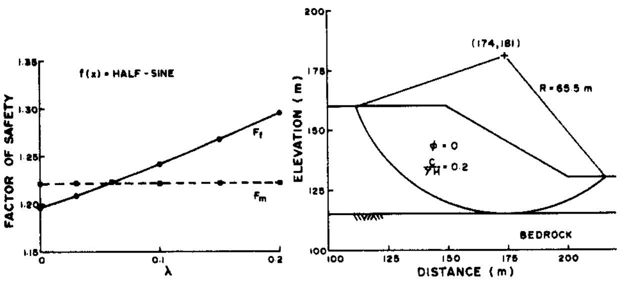

Fig. 2 Comparison of Force and Moment Factors of Safety for Ø = 0 Analysis

Bishop and Morgenstern (1960) identified the variables for the stability assessment of a simple slope (Figure 3). These variables are as follows: Ø' = effective angle of friction; c' = effective cohesion intercept; γ = total unit weight of soil; r_u = pore pressure ratio; H = slope height; β = slope inclination; and D = depth factor of a hard stratum. The parameters c', γ, H can be combined to form the dimensionless parameter c'/γH. Table 2 shows the values for the variables (i.e., Ø', c'/γH, r_u, D, β) used in this study.

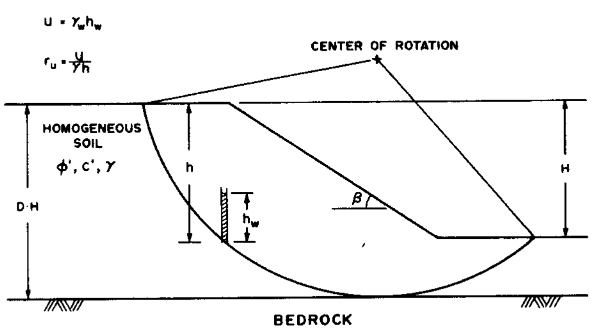

Fig. 3 Specifications of Variables for a Simple Slope

TABLE II

SUMMARY OF VALUES OF THE VARIABLES CONSIDERED

Variable	Value
Ø'	20°, 30°, 40°
c'/γH	0, 0.0125, 0.025, 0.05, 0.1, 0.2
r_u	0, 0.25, 0.5
D	0.00*, 1.00, 1.25, 1.50
β	10°, 20°, 30°, 45°⁺, 60°⁺

* D = 0.00 used to designate toe circles

\+ Toe circles only.

The method of analysis can be regarded as an independent variable since the main objective of this study was to compare the factors of safety calculated by these methods. Comparisons were made for the Ordinary, the Bishop's Simplified, the Janbu's Simplified, the Janbu's Generalized, the Spencer and the Morgenstern-Price methods. A half sine interslice force function was used on the Morgenstern-Price method. Results obtained from the Bishop's Simplified method were used as a basis of comparison in this study. The Lowe-Karafiath and Corps of Engineers methods were not included.

All calculations were performed using a circular slip surface through simple, homogeneous slopes. Comparisons were made on the critical slip surfaces for each method.

A convergence tolerance of 0.001 was used in the computations of the factors of safety. A larger tolerance limit of 0.002 to 0.01 was used when convergence difficulties were encountered. The position for the critical center was determined to within a distance of plus or minus 0.025H using an automatic search routine in SLOPE-II. The width of slices was chosen such that a minimum of 30 slices would be inscribed within any specified slip surface.

PRESENTATION AND ANALYSIS OF RESULTS

The stability coefficients m and n (Bishop and Morgenstern, 1960) are used for the presentation of data.

$$F = m - nr_u \quad (18)$$

where m and n = parameters corresponding to the intercept and slope of the F versus r_u plot, respectively.

The presentation can be reduced using these coefficients and the influence of the pore-water pressure can be readily visualized.

For c'/γH = 0, all methods of slices give identical results. Comparisons for such case have not been included.

The m and n values for toe circles are compared for c'/γH equal 0.0125 (Figure 4) since the critical surface is shallow for highly frictional soil slopes. The m values calculated by the Ordinary method are always slightly smaller than those computed by Bishop's Simplified method. The differences in the m values decrease with the decrease in slope inclinations and decrease with the increase in friction angles. The differences are smaller than 6%. The n values computed by the Ordinary method are usually higher for flatter slopes and lower for steeper slopes as compared to those obtained from Bishop's Simplified method. The differences are less than 10%.

For Janbu's Simplified method with a correction factor, the computed m and n values are generally higher. The differences in the m and n values are approximately 5 and 10%, respectively. The m and n values calculated by Bishop's Simplified, Janbu's Generalized, Spencer and Morgenstern-Price methods are identical. When using Janbu's Generalized, Spencer and Morgenstern-Price methods to analyze slopes

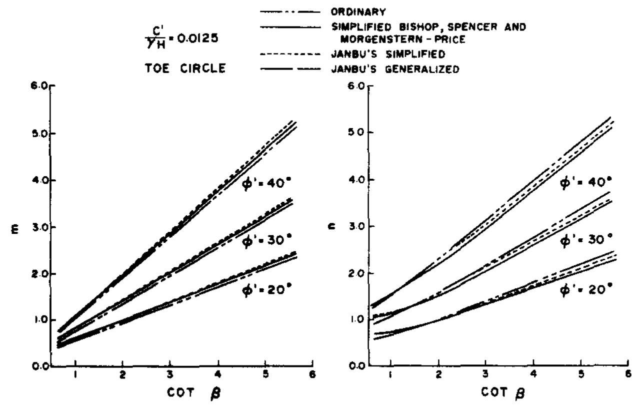

Fig. 4 Comparisons of m and n Values for D = 0.0 and $c'/\gamma H$ = 0.0125

steeper than 30 degrees, convergence problems were common (Ching, 1981). Higher tolerance limits used to achieve convergence resulted in slightly wider deviations in the m and n values.

Results for base circles with D = 1.00 and $c'/\gamma H$ = 0.025 are compared (Figure 5). Investigations for base circles have not been conducted for slopes exceeding 30 degrees since the critical slip surface invariably passes through the toe of the slope. For the Ordinary method, the m values are consistently lower while the n values are higher than those calculated by Bishop's Simplified method. The difference in the computed m and n values range from 2% for flatter slopes to 5% for steeper slopes. For Janbu's Simplified method, the m and n values are slightly greater than those obtained from Bishop's Simplified method. Their differences are about 1 to 3%. Again, the m and n values calculated by other methods are identical.

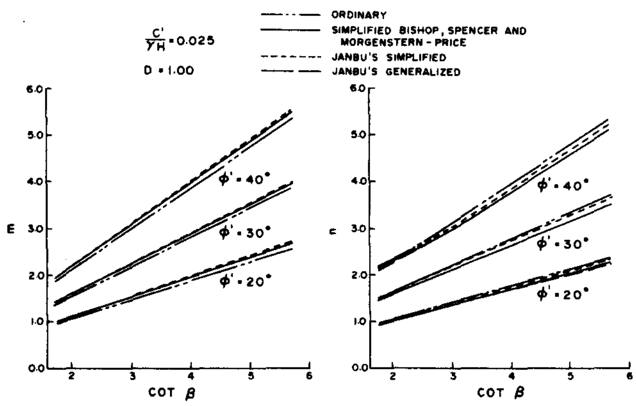

Fig. 5 Comparisons of m and n Values for D = 1.00 and $c'/\gamma H$ = 0.025

As the soil slope becomes progressively more cohesive, the critical slip circle tends to penetrate below the toe. Therefore, results for base circles with D = 1.25 are compared for $c'/\gamma H$ equal to 0.025 and 0.05 (Figures 6 and 7). The Ordinary method consistently gives lower m

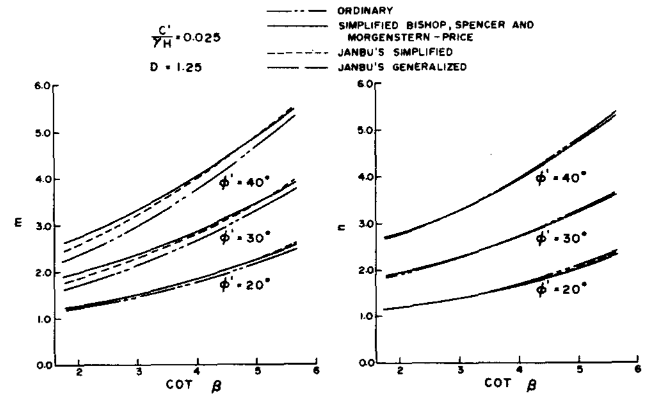

Fig. 6 Comparisons of m and n Values for D = 1.25 and $c'/\gamma H$ = 0.025

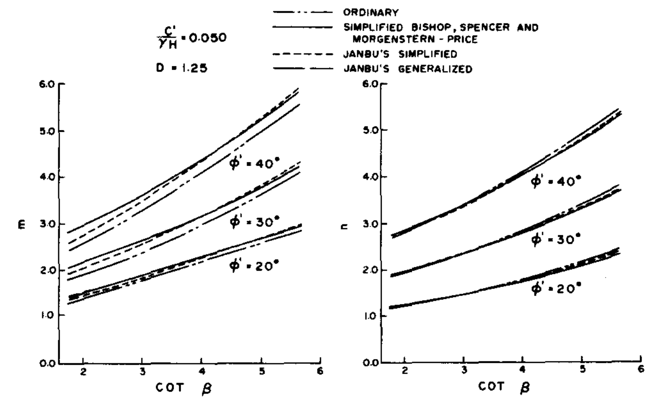

Fig. 7 Comparisons of m and n Values for D = 1.25 and $c'/\gamma H$ = 0.05

values than those obtained from Bishop's Simplified method. Its computed n values are always lower for slopes steeper than 20 degrees but higher for slopes flatter than 20 degrees as compared with those calculated by Bishop's Simplified method. The differences in the m values which increase with the slope inclination generally range from 3 to 15%. The difference in the n values remain at about 4%. Janbu's Simplified method yields lower m values for steeper slopes but higher m values for flatter slopes when compared to Bishop's Simplified method. The differences are about 1% for a 10-degree slope and 8% for a 30-degree slope. The computed n values are usually higher for steeper slopes and lower for flatter slopes. The differences generally do not exceed 4%. Other methods of slices give comparable results for both m and n values with differences less than 1%.

The m and n values for base circles with D = 1.50 are compared for $c'/\gamma H$ equal to 0.05 and 0.20 (Figures 8 and 9). For the Ordinary method, the computed m values are substantially smaller than those calculated by Bishop's Simplified method. The differences range from 5 to 20%

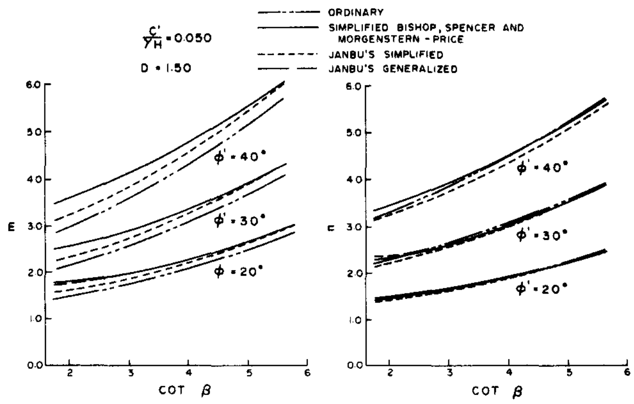

Fig. 8 Comparisons of m and n Values for $D = 1.50$ and $c'/\gamma H = 0.05$

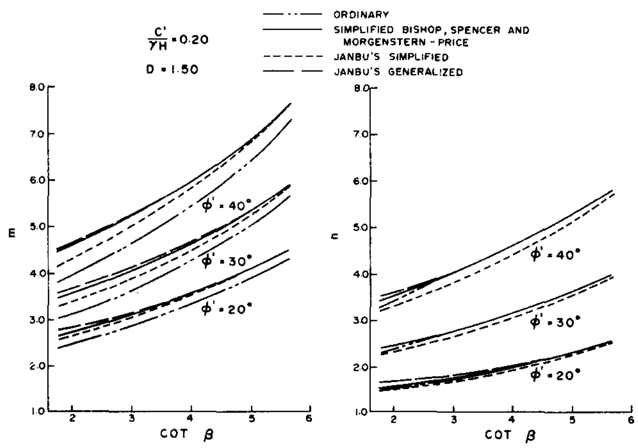

Fig. 9 Comparisons of m and n Values for $D = 1.50$ and $c'/\gamma H = 0.2$

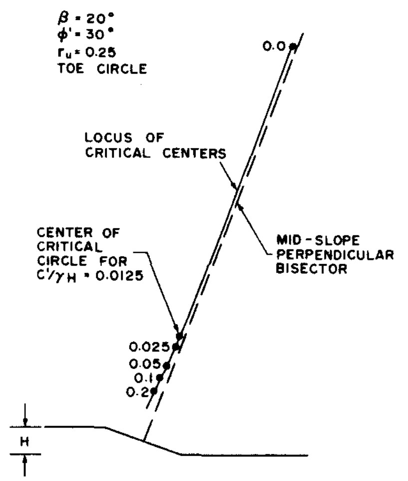

Fig. 10 Variation in Critical Centres with $c'/\gamma H$ ---- Bishop's Simplified Method

and increase with the slope angle. The computed n values are generally smaller for steeper slopes but higher for flatter slopes. The differences in the n values are approximately 5%. For Janbu's Simplified method, the computed m values are intermediate between those calculated by the Ordinary and Bishop's Simplified methods. The computed n values, however, are usually smaller than those obtained from Bishop's Simplified method. Variations in the m and n values are about 1 to 10%. All other methods give results close to those calculated by Bishop's Simplified method. The differences in the m values are about 1% and the differences in the n values are about 3%.

The position of the critical slip circle and its center of rotation vary with respect to the values of $c'/\gamma H$, r_u, β and D. In general, the centre tends to drop downward along the bisecting line perpendicular to the slope face, as the value of $c'/\gamma H$ increases (Figure 10). The critical centre moves along the mid-slope normal line away from the slope as the friction angle increases (Figure 11). Variations in the critical centre with respect to the pore pressure ratio are similar to reducing the angle of internal friction of the soil. For highly frictional soil slopes, the critical circle invariably passes through the toe of the slope regardless of the values of $c'/\gamma H$ and β. When the cohesive strength of the soil dominates, the critical slip circle tends to penetrate to a greater depth, particularly for flatter slopes.

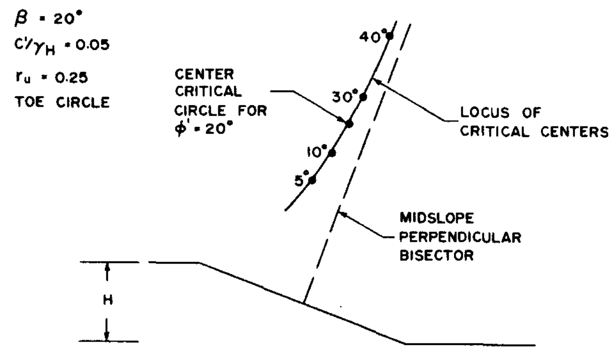

Fig. 11 Variation in Critical Centres with ϕ' ---- Bishop's Simplified Method

CONCLUSIONS

Based on this study, the following conclusions have been drawn.

1. Two special cases of slope stability analysis are presented. In the special case when $c' = 0$, all methods of slices give identical factors of safety. In the special case when $\phi = 0$, methods which satisfy moment or moment and force equilibrium give identical factors of safety. Force equilibrium methods give variable results depending on the side force assumptions.
2. For shallow slip circles, different methods give comparable results. The differences in the m and n values obtained from various

methods are usually small. Differences are less than 10%.

3. For deeper slip circles, different methods give variable results. The differences in the m and n values obtained from the Ordinary and Bishop's Simplified methods range from 5 to 20%. The differences in the m and n values from Janbu's Simplified and Bishop's Simplified methods are also quite large. They are in the order of 5 to 10%. Other methods give results comparable to Bishop's Simplified method with differences less than 5%. The differences in the m and n values from various methods increase with the soil friction angle and slope inclination. However, deep-seated slip surface is not likely to be critical and the critical surface is a shallow surface passing through the toe of the slope.

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