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Coupled Three-Dimensional Consolidation Theory of Unsaturated Porous Media

V. DAKSHANAMURTHY
Senior Geotechnical Engineer, R & M Consultants Inc., Alaska, USA
D.G. FREDLUND
Professor of Civil Engineering, University of Saskatchewan, Canada
and
H. RAHARDJO
Research Engineer in Civil Engineering, University of Saskatchewan, Canada

SUMMARY A three-dimensional, two-phase flow, consolidation theory is presented for unsaturated porous media. The continuity and equilibrium conditions are coupled in deriving the consolidation equations. Their application to a one-dimensional case is illustrated.

1 INTRODUCTION

The application of a load to a soil will cause the soil to undergo a time dependent volume change. This transient process is known as consolidation. The first theory of consolidation was first derived by Terzaghi in the 1930's for saturated soils.

During the past few years, there has been a considerable increase in the understanding of the behavior of unsaturated soils. Several consolidation theories for unsaturated soils have been suggested. This paper presents a coupled, two-phase flow, consolidation theory for unsaturated soils. A three-dimensional consolidation is first derived which is then specialized to a one-dimensional case. Basic physical and constitutive relations required for the derivations are introduced. Elastic equilibrium and continuity criteria are coupled and used in the derivations. The formulation is somewhat similar to that proposed by Biot in 1941; however, the unsaturated soil terminology has changed considerably.

2 NOTATION

The following symbols are used in this paper:

c_v^a = the coefficient of consolidation with respect to the air phase
C_a = the interactive constant associated with the air phase equation
c_g = the gravity term constant
c_v^w = the coefficient of consolidation with respect to the water phase
C_w = the interactive constant associated with the water phase equation
D = a transmission constant for the air phase having the same units as coefficient of permeability
D* = the transmission constant of proportionality for the air phase
E_1 = Young's modulus for the soil structure with respect to $(\sigma - u_a)$
g = the gravitational acceleration
G = the shear bulk modulus
h_a = the total head in the air phase
h_w = the total head in the water phase
H_1 = the elastic modulus for the soil structure with respect to $(u_a - u_w)$
H_1' = the elastic modulus for the water phase with respect to $(\sigma - u_a)$
ε = volumetric strain of the soil structure
H_1'' = the elastic modulus for the air phase with respect to $(\sigma - u_a)$
k_w = the coefficient of permeability with respect to the water phase
m = the mass of the air phase
n = the porosity of the soil
p = the absolute pore-air pressure
R = the universal gas constant
R_1 = the elastic modulus for the water phase with respect to $(u_a - u_w)$
R_1' = the elastic modulus for the air phase with respect to $(u_a - u_w)$
S = the degree of saturation of the soil
t = time
u = the displacement in the x-direction
u_a = the pore-air pressure
u_{atm} = the atmospheric pressure
u_w = the pore-water pressure
$(u_a - u_w)$ = the matric suction
v = the displacement in the y-direction
v_w = the velocity of water
w = the displacement in the z-direction
Y = the elevation head from an arbitrary datum
$\gamma_{xy}, \gamma_{yz}, \gamma_{xz}$ = the shearing strains in the xy-, yz- and xz-planes, respectively.
$\varepsilon_x, \varepsilon_y, \varepsilon_z$ = the normal strains in the x-, y-, and z-directions, respectively.
θ = the absolute temperature
θ_a = the change in volume of air in the element due to flow or compression
θ_w = the net inflow or outflow of water for the element
μ = Poisson's ratio
ρ_a = the density of air
ρ_w = the density of water
σ = the normal stress
$\sigma_x, \sigma_y, \sigma_z$ = the normal stress in the x-, y-, and z-directions, respectively.
$\tau_{xy}, \tau_{yz}, \tau_{xz}$ = the shearing stress in the xy-, yz- and xz-planes, respectively.
ω = the molecular weight of the mass of air

3 UNSATURATED SOILS

The effective stress concept for saturated soils (Terzaghi, 1936) has been widely examined and accepted. The stress state variable for saturated soils is defined by $(\sigma - u_w)$ and is called the effective stress. Bishop (1959) proposed a single-valued equation to describe the state of stress for unsaturated soils. However, the use of two independent stress tensors for unsaturated soils has proven to be more satisfactory (Fredlund and Morgenstern, 1977).

3.1 Stress State Variables

In 1977, Fredlund and Morgenstern postulated that an element of an unsaturated soil can be visualized as a four phase system. Under an applied stress gradient, two of the phases will come to equilibrium (i.e., soil particles and contractile skin or air-water interface) whereas the other two phases will flow (i.e., air and water). Various combinations of independent stress state variables are possible according to the theoretical analysis based on the multiphase continuum mechanics. However, the $(\sigma - u_a)$ and $(u_a - u_w)$ combination is advantageous because the effects of changes in total stress and pore-water pressure can be separated. In addition, the pore-air pressure is most often atmospheric (i.e., $u_a = 0$). Two independent tensors are used to describe the stress state for an unsaturated soil (i.e., three-dimensional). The first tensor is,

$$\begin{bmatrix} (\sigma_x - u_a) & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & (\sigma_y - u_a) & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & (\sigma_z - u_a) \end{bmatrix}$$

and the second tensor is,

$$\begin{bmatrix} (u_a - u_w) & 0 & 0 \\ 0 & (u_a - u_w) & 0 \\ 0 & 0 & (u_a - u_w) \end{bmatrix}$$

A smooth transition in the stress state variables can be obtained when going from the unsaturated to the saturated soil case. The pore-air pressure and the pore-water pressure become approximately equal in magnitude as the degree of saturation approaches 100%. Therefore, the matric suction term, $(u_a - u_w)$, goes to zero and the $(\sigma - u_a)$ term becomes $(\sigma - u_w)$.

3.2 Flow Laws

Certain physical relations are required for deriving the partial differential flow equations for air and water phases.

3.2.1 Water phase

Darcy's Law is commonly used to describe the flow of water in a saturated soil. The same law also applies for the flow of water through an unsaturated soil (Childs and Collis-George, 1950; Freeze and Cherry, 1979).

$$v_w = -k_w \frac{\partial h_w}{\partial y} \qquad (1)$$

$$h_w = Y + \frac{u_w}{g \rho_w} \qquad (2)$$

Note that the velocity head is neglected.

3.2.2 Air phase

The flow of air through an unsaturated soil can be described using Fick's Law (Blight, 1971).

$$v_a = -D^* \frac{\partial u_a}{\partial y} \qquad (3)$$

where:

$$D^* = D/g$$

$$h_a = \frac{u_a}{g \rho_a} \qquad (4)$$

Note that the velocity and elevation heads are negligible.

The density of air under isothermal condition can be written as follows:

$$\rho_a = \left(\frac{\omega}{R \theta g}\right) p \qquad (5)$$

where: $p = u_a + u_{atm}$

The mass of air can be computed as,

$$m = (1 - S)\, n\, \rho_a\, g \qquad (6)$$

3.3 Constitutive Relations

The stress and deformation state variables can be linked by suitable constitutive relations which incorporate material properties. The proposed constitutive equations are similar to those suggested by Biot (1941) and Coleman (1962).

The continuity of a referential, unsaturated soil element requires that the overall (or the soil structure) volume change of the element to be equal to the sum of the volume changes associated with each phase. Assuming the soil particles are incompressible and considering the volume change of the contractile skin to be internal to the element, Fredlund and Morgenstern (1976) proposed the continuity requirement as follows:

$$\epsilon = \theta_w + \theta_a \qquad (7)$$

Therefore, it is necessary to establish two constitutive equations to describe the volume change behavior in unsaturated soils.

3.3.1 Soil structure

The soil is assumed to behave as an isotropic and linear elastic material. This assumption is acceptable in an incremental sense. The constitutive relations can be developed in a semi-empirical manner as an extension of the elasticity formulation used for saturated soils.

$$\epsilon_x = \frac{(\sigma_x - u_a)}{E_1} - \frac{\nu}{E_1}(\sigma_y + \sigma_z - 2u_a) + \frac{(u_a - u_w)}{H_1} \qquad (8)$$

$$\epsilon_y = \frac{(\sigma_y - u_a)}{E_1} - \frac{\nu}{E_1}(\sigma_x + \sigma_z - 2u_a) + \frac{(u_a - u_w)}{H_1} \qquad (9)$$

$$\epsilon_z = \frac{(\sigma_z - u_a)}{E_1} - \frac{\nu}{E_1}(\sigma_x + \sigma_y - 2u_a) + \frac{(u_a - u_w)}{H_1} \qquad (10)$$

$$\gamma_{xy} = \tau_{xy}/G \qquad (11)$$

$$\gamma_{yz} = \tau_{yz}/G \qquad (12)$$

$$\gamma_{xz} = \tau_{xz}/G \quad (13)$$

The volumetric strain, ϵ, can be obtained from the summation of the normal strains as follows:

$$\epsilon = \epsilon_x + \epsilon_y + \epsilon_z \quad (14)$$

Rearranging (8), (9), (10), the normal stresses can be expressed as functions of the normal strains and matric suction.

$$(\sigma_x - u_a) = 2G(\epsilon_x + \alpha\epsilon) - \beta(u_a - u_w) \quad (15)$$

$$(\sigma_y - u_a) = 2G(\epsilon_y + \alpha\epsilon) - \beta(u_a - u_w) \quad (16)$$

$$(\sigma_z - u_a) = 2G(\epsilon_z + \alpha\epsilon) - \beta(u_a - u_w) \quad (17)$$

where:

$$G = \frac{E_1}{2(1+\mu)}$$

$$\alpha = \left(\frac{\mu}{1-2\mu}\right)$$

$$\beta = \frac{E_1}{H_1}\frac{1}{(1-2\mu)} \quad \text{or} \quad \frac{2G}{H_1}\frac{(1+\mu)}{(1-2\mu)}$$

The shearing stresses from (11), (12) and (13) can be written as follows:

$$\tau_{xy} = G\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \quad (18)$$

$$\tau_{yz} = G\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) \quad (19)$$

$$\tau_{xz} = G\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) \quad (20)$$

3.3.2 Water phase

The constitutive relations for the water phase defines the volume of water in the element for any combination of total, pore-air and pore-water pressures. The net inflow or outflow of water from a referential type element can be written as a linear combination of the effects of each stress state variable (Biot, 1941; Coleman, 1962; Fredlund and Morgenstern, 1976).

$$\theta_w = \frac{(\sigma_x + \sigma_y + \sigma_z - 3u_a)}{3H_1'} + \frac{(u_a - u_w)}{R_1} \quad (21)$$

Substituting (15), (16) and (17) into (21), the net flow of water can be rewritten as,

$$\theta_w = \frac{\beta}{3}\epsilon + \gamma(u_a - u_w) \quad (22)$$

where: $\gamma = \frac{1}{R_1} - \frac{\beta}{H_1'}$

3.3.3 Air phase

The constitutive relations for the air phase defines the volume of air in the element for any combination of the total, pore-air and pore-water pressures. Similarly, the net inflow or outflow of air can be written as a linear combination of the effects of each stress state variable.

$$\theta_a = \frac{(\sigma_x + \sigma_y + \sigma_z - 3u_a)}{3H_1''} + \frac{(u_a - u_w)}{R_1'} \quad (23)$$

Substituting (15), (16) and (17) into (23), the net flow of air, θ_a, can be written in a different form.

$$\theta_a = \frac{\beta}{3}\epsilon + \gamma'(u_a - u_w) \quad (24)$$

where: $\gamma' = \frac{1}{R_1'} - \frac{\beta}{H_1''}$

According to the volumetric continuity requirement (7), there is a fixed relationship between the soil structure (or overall), water and air phase moduli.

4 DERIVATIONS OF COUPLED CONSOLIDATION EQUATIONS

The consolidation of an unsaturated soil can be treated as a transient flow problem with water and air flow independently. Therefore, two independent partial differential equations are required to solve for the pore-water and pore-air pressure that changes with time. Both equations must satisfy the continuity of the water and air phases, respectively. This method is called a two-phase flow approach and has been presented by Fredlund and Hasan (1979), Lloret and Alonso (1980) and Fredlund (1982).

For a rigorous formulation of two- and three-dimensional consolidation, the continuity equations should be coupled with the equilibrium equations. This method was proposed by Biot (1941) to analyze the consolidation process for a special case of an unsaturated soil. The derivations were based on the assumption of occluded bubbles of air during the consolidation process.

The coupled consolidation equations presented in this paper assume that the air phase is continuous. Several other assumptions used in the derivation will be similar to those proposed by Terzaghi (1943) and Biot (1941) with the outline as follows: (1) material is isotropic, (2) reversibility of stress-strain relations, (3) linearity of stress-strain relations, (4) small strains, (5) pore-water is incompressible, (6) coefficients of permeabilities of water and air phases are functions of the volume-mass soil properties during the consolidation process, (7) the effect of air diffusing through water, air dissolving in water phase and the movement of water vapor are ignored.

4.1 Three-Dimensional Case

Let us consider a referential element of an unsaturated soil.

4.1.1 Equilibrium equations

The stress state for an unsaturated soil element should satisfy the following equilibrium conditions:

$$\begin{aligned} \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} &= 0 \\ \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} &= 0 \\ \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} &= 0 \end{aligned} \quad (25)$$

Substituting (15), (16) and (17) into (25) and rearranging, equation (26) can be obtained.

$$G\nabla^2 u + \frac{G}{1-2\mu}\frac{\partial \epsilon}{\partial x} - \beta\frac{\partial(u_a - u_w)}{\partial x} + \frac{\partial u_a}{\partial x} = 0$$

$$G\nabla^2 v + \frac{G}{1-2\nu}\frac{\partial\epsilon}{\partial y} - \beta\frac{\partial(u_a - u_w)}{\partial y} + \frac{\partial u_a}{\partial y} = 0$$

$$G\nabla^2 w + \frac{G}{1-2\nu}\frac{\partial\epsilon}{\partial z} - \beta\frac{\partial(u_a - u_w)}{\partial z} + \frac{\partial u_a}{\partial z} = 0 \qquad (26)$$

where: $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial y^2}$

4.1.2 Water phase continuity

During the consolidation process, water flows due to a hydraulic gradient. The net flux of water per unit volume of the element can be computed using Darcy's Law (1).

$$\frac{\partial\theta_w}{\partial t} = -\left[\frac{k_w}{\rho_w g}\nabla^2 u_w + \frac{1}{\rho_w g}\left(\frac{\partial k_w}{\partial x}\frac{\partial u_w}{\partial x} + \frac{\partial k_w}{\partial y}\frac{\partial u_w}{\partial y} + \frac{\partial k_w}{\partial z}\frac{\partial u_w}{\partial z}\right) + \frac{\partial k_w}{\partial y}\right] \qquad (27)$$

The coefficients of permeability in x-, y- and z-directions are the same (k_w) due to the assumption of isotropy. However, (27) assumes that the coefficient of permeability can vary significantly with space for the unsaturated soil.

Differentiating the constitutive equation for the water phase, (22), with respect to time and equating it with (27) the continuity equation for the water phase can be obtained.

$$-\left[\frac{k_w}{\rho_w g}\nabla^2 u_w + \frac{1}{\rho_w g}\left(\frac{\partial k_w}{\partial x}\frac{\partial u_w}{\partial x} + \frac{\partial k_w}{\partial y}\frac{\partial u_w}{\partial y} + \frac{\partial k_w}{\partial z}\frac{\partial u_w}{\partial z}\right) + \frac{\partial k_w}{\partial y}\right] = \frac{\beta}{3}\frac{\partial\epsilon}{\partial t} + \gamma\frac{\partial(u_a - u_w)}{\partial t} \qquad (28)$$

4.1.3 Air phase continuity

The air phase is compressible and flows in response to an air pressure gradient. The net mass flux of air per unit volume of the element is obtained according to Fick's law.

$$\frac{\partial m}{\partial t} = -D^* \nabla^2 p \qquad (29)$$

The change in coefficient of permeability with respect to space is assumed to be negligible as indicated in (29). The net volume flux of air per unit volume of the element can be computed by differentiating the mass and volume relationship of air, together with using (5) and (6).

$$\frac{\partial\theta_a}{\partial t} = \frac{\partial(m/\rho_a g)}{\partial t} = -\frac{1}{\rho_a g} D^* \nabla^2 u_a - \frac{(1-S)\,n}{(u_a + u_{atm})}\frac{\partial u_a}{\partial t} \qquad (30)$$

The constitutive equation for the air phase, (24), is differentiated with respect to time and equated with (30) to formulate the continuity equation for the air phase.

$$-\left[\frac{D^*}{\rho_a g}\nabla^2 u_a + \frac{(1-S)\,n}{(u_a + u_{atm})}\frac{\partial u_a}{\partial t}\right] = \frac{\beta}{3}\frac{\partial\epsilon}{\partial t} + \gamma'\frac{\partial(u_a - u_w)}{\partial t} \qquad (31)$$

4.2 Application to a One-Dimensional Case

Let us consider a column of unsaturated soil. This column of soil is subjected to a one-dimensional consolidation with the following conditions: An external load is applied in the y-direction while the column is confined laterally (i.e., $u = w = 0$). As a result, a vertical stress, σ_y, is developed and assumed to be constant with time. The water and air phases are assumed to flow only in the vertical direction (i.e., the y-direction). Therefore, the vertical displacement, v, and the net flux of water, θ_w, and of air, θ_a, are only functions of the y-coordinate and time, t.

4.2.1 Equilibrium equation

In a one-dimensional case, where there is no lateral deformation (i.e., $u = w = 0$), the three equilibrium equations, (26), revert to a single equilibrium equation as follows:

$$2G\frac{(1-\nu)}{(1-2\nu)}\frac{\partial^2 v}{\partial y^2} + \beta\frac{\partial u_w}{\partial y} - (\beta - 1)\frac{\partial u_a}{\partial y} = 0 \qquad (32)$$

Integrating (32) along the y-axis, and differentiating it with respect to time, t, gives,

$$\frac{\partial^2 v}{\partial y\,\partial t} = \frac{(\beta-1)\frac{\partial u_a}{\partial t} - \beta\frac{\partial u_w}{\partial t}}{2G\frac{(1-\nu)}{(1-2\nu)}} \qquad (33)$$

4.2.2 Water phase partial differential equation

Considering the assumption that the water phase is only dependent on the y-coordinate and time, t, the continuity equation, (28), can be rewritten for one-dimensional consolidation as follows:

$$-\frac{k_w}{\rho_w g}\frac{\partial^2 u_w}{\partial y^2} = \frac{\beta}{3}\frac{\partial^2 v}{\partial y\,\partial t} + \gamma\frac{\partial u_a}{\partial t} - \gamma\frac{\partial u_w}{\partial t} + \frac{1}{\rho_w g}\frac{\partial k_w}{\partial y}\frac{\partial u_w}{\partial y} + \frac{\partial k_w}{\partial y} \qquad (34)$$

By substituting the equilibrium equation, (33), into the continuity of the water phase, (34), the water phase partial differential equation can be derived.

$$\frac{\partial u_w}{\partial t} = -C_w\frac{\partial u_a}{\partial t} + c_v^w\frac{\partial^2 u_w}{\partial y^2} + \frac{c_v^w}{k_w}\frac{\partial k_w}{\partial y}\frac{\partial u_w}{\partial y} + c_g\frac{\partial k_w}{\partial y} \qquad (35)$$

where:

$$C_w = (1 - m_2^w/m_1^w)/(m_2^w/m_1^w)$$

$$c_v^w = k_w/\rho_w g\, m_2^w$$

$$m_1^w = \frac{1}{3H_1}\frac{(1+\nu)}{(1-\nu)}$$

$$m_2^w = \frac{E_1}{H_1(1-2\nu)}\left[\frac{1}{3H_1}\frac{(1+\nu)}{(1-\nu)} - \frac{1}{H_1'}\right] + \frac{1}{R_1}$$

$$c_g = 1/m_2^w$$

4.2.3 Air phase partial differential equation

Similar to the water phase, the continuity equation for the air phase, (31), can be rewritten for a one-dimensional consolidation as follows:

$$-\frac{D^*}{\rho_a g}\frac{\partial^2 u_a}{\partial y^2} = \frac{\beta}{3}\frac{\partial^2 v}{\partial y\,\partial t} + \left(\gamma' + \frac{(1-S)n}{u_a + u_{atm}}\right)\frac{\partial u_a}{\partial t} - \gamma'\frac{\partial u_w}{\partial t} \quad (36)$$

Substituting the equilibrium equation, (33), into the continuity of the air phase, (36), gives the air phase partial differential equation

$$(1 - m_2^a/m_1^a) - \frac{(1-S)n}{(u_a + u_{atm})m_1^a}$$

$$c_v^a = \frac{D^* R\theta}{\omega}\;\frac{1}{(1 - m_2^a/m_1^a)(u_a + u_{atm})\,m_1^a + (1-S)\,n}$$

$$m_1^a = \frac{1}{3H_1}\,\frac{(1+\mu)}{(1-\mu)}$$

$$m_2^a = \frac{E_1}{H_1(1-2\mu)}\left[\frac{1}{3H_1}\,\frac{(1+\mu)}{(1-\mu)} - \frac{1}{H_1''}\right] + \frac{1}{R_1}$$

The change in pore-air and pore-water pressures is computed by solving (35) and (37) simultaneously. As a soil becomes saturated, the interactive constant, C_w, approaches zero. If the variation in coefficient of permeability with respect to space is negligible, (35) reverts to Terzaghi's one-dimensional consolidation for saturated soils. As a soil becomes completely dry, the interactive constant, C_a, approaches zero, and (37) reverts to the form presented by Blight (1971). The overall (i.e., soil structure) deformation is obtained from (33).

5 CONCLUSIONS

The theory of consolidation for an unsaturated porous media is formulated by coupling the continuity and the equilibrium requirements. The solution is obtained by solving the partial differential equations for the water and air phases simultaneously. The proposed coupled transient flow model provides a smooth conceptual transition from the unsaturated to the saturated case.

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