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STEADY STATE AND TRANSIENT MASS TRANSPORT MODELS FOR SATURATED - UNSATURATED SOILS

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ABSTRACT

A steady state and a transient model are presented describing the hydraulic head flow component of mass transport through saturated-unsaturated soils. Continuous flow is considered between saturated and unsaturated soil zones. For unsaturated soils, the coefficient of permeability is considered as a function of the negative pressure head. The differential equation governing the flow is solved using an iterative finite element scheme. The steady-state and the transient models are implemented into the computer programs SEEP and TRASEE, respectively. Two example problems are presented to demonstrate the capabilities of the models.

KEYWORDS

Steady state, transient, finite element models, water flow, saturated-unsaturated soils.

INTRODUCTION

Groundwater contamination from industrial wastes is of major environmental concern. There is an increasing number of engineering problems related to the control of pollutant movement through soils. The problems encountered range from seepage calculation through dams and dykes to the control of groundwater contamination from tailings disposal leachates. The solution of these problems awaits the development of rigorous analytical models simulating the problems at hand.

There are two major mechanisms contributing to the transport of contaminants through groundwater systems, namely advection and dispersion. Advection flow is the mass transport component occurring in response to hydraulic head gradients. Dispersion refers to the spreading of pollutants caused by molecular diffusion and microscopic variations in velocities within the soil pores. For most field problems, the dispersive component of the transport process can be considered negligible compared with advection. In such cases, pollutant concentrations can be estimated from the velocity field calculations considering only the advection mechanism.

Two models are proposed describing the advection flow component of pollutant transport through saturated-unsaturated soils. The models deal with two dimensional steady state and transient problems. The models are formulated and computer implemented into finite element computer programs called SEEP and TRASEE. Two example problems are presented to demonstrate the capabilities of the developed software.

BACKGROUND

Some of the earliest theoretical work in the area of flow through unsaturated soils was presented by Richards in 1931. Richards (1931) recognized that "the essential difference between flow through a porous medium which is saturated and flow through medium which is unsaturated is that under the latter condition the pressure is described by capillary forces and the conductivity depends on the moisture content of the medium".

In 1937, Casagrande proposed a graphical method for the solution of seepage problems. The flow net technique considered the flow of water in the saturated zone only. To overcome the difficulty associated with flow through saturated-unsaturated soils, flow problems are separated into confined and unconfined cases. In the case of unconfined flow, the upper boundary of the flow region, referred to as the line of seepage, is unknown and must be empirically determined in order to draw a flow net.

In 1958, Gardner proposed a general relationship between the coefficient of permeability and the capillary pressure for unsaturated soils. The proposed relationship is given by the following equation:

$$k = \frac{a}{b + p_c^n} \tag{1}$$

where: k = intrinsic or material permeability, (cm^2)
p_c = capillary pressure, ($dynes/cm^2$)
n = positive dimensionless constant
a,b = constants depending upon the system of units used.

Taylor and Brown (1967) proposed a finite element model for seepage problems with "a free surface". This model considered the flow of water in the saturated zone only and as a result, "the principal problem is locating the position of the free surface that has both zero flow normal to it and prescribed pressure". The proposed trial and error procedure for locating the free surface is cumbersome and often results in convergence problems.

In 1971, Freeze presented a three-dimensional finite difference model for transient flow through saturated-unsaturated soil. The model "treats the complete subsurface region as a unified whole considering continuous flow between the saturated and unsaturated zones". Criticizing the saturated-only models, Freeze (1971) stated that "boundary conditions that are satisfied on the free surface specify that the pressure head must be atmospheric and the surface must be a streamline. Whereas the first of these conditions is true, the second is not". With respect to the continuous saturated-unsaturated flow approach, Freeze (1971) stated "we can avoid the incorrect boundary conditions by solving the complete saturated-unsaturated boundary value problem".

For the solution of steady state flow problems, engineers have traditionally relied on the flow net technique (Casagrande, 1937). Flow models adopted later described flow in the saturated zone only (eg., Taylor and Brown, 1967). This "loyalty" to the concepts of confined and unconfined flow along with a lack of understanding on the behaviour of unsaturated soils have discouraged the use of models considering continuous flow in saturated-unsaturated systems. Recent developments in the area of unsaturated soils (Fredlund and Morgenstern, 1976 and Daksanamurthy and Fredlund, 1980), offer the necessary background for using comprehensive flow models for saturated-unsaturated soils.

In 1982, Papagianakis proposed a two-dimensional finite element model for flow in saturated-unsaturated soils. A computer program, SEEP, was developed for steady state flow conditions. Lam (1983) extended the research to a general steady and transient state finite element flow model called TRASEE.

THEORY AND FINITE ELEMENT FORMULATION

The proposed model describes the flow of water in a two-dimensional saturated-unsaturated soil systems. In the unsaturated zone, the air phase is assumed continuous and under atmospheric pressure. Therefore, the model considers only the flow of water. Water flows in response to hydraulic head gradients, regardless of the degree of saturation of the soil. For both saturated and unsaturated soil, the water discharge velocity is assumed to be defined by Darcy's law. In the unsaturated soil, it is assumed that the coefficients of permeability of the medium depends on the negative pressure head.

The differential equation governing the flow is derived by equating the net flow quantity from a soil element to the time derivative of the constitutive relationship for an unsaturated soil (Fredlund and Morgenstern, 1976).

$$\frac{\partial}{\partial x}\left[k_{xx}\frac{\partial h}{\partial x} + k_{xy}\frac{\partial h}{\partial y}\right] + \frac{\partial}{\partial y}\left[k_{yx}\frac{\partial h}{\partial x} + k_{yy}\frac{\partial h}{\partial y}\right] = \rho_w g m_2^w \frac{\partial h}{\partial t} \qquad (2)$$

where: k_{xx}, k_{xy}, k_{yx}, k_{yy} = permeability tensor of an anisotropic soil
h = total head
m_2^w = slope of the $(u_a - u_w)$ versus Θw plot when $d(\sigma - u_a) = 0$
u_a = pore - air pressure
u_w = pore - water pressure
Θw = net inflow or outflow of water for the element
σ = total stress
ρ_w = density of water
g = acceleration of gravity

For steady state conditions, the right-hand side of equation (2) goes to zero. Equation 2 is nonlinear because the values of the coefficient of permeability are not constant but depend upon the unknown pressure head. Using the Galerkin principle of weighted-residuals, the flow equation for an element can be expressed as follows:

$$\int_A \{B\}^T[K]\{B\}dA\{h^n\} + \int_A \{L\}^T \lambda \{L\}\frac{\partial\{h^n\}}{\partial t}\, dA - \int_S \{L\}^T q\, ds = 0 \qquad (3)$$

where: $\{B\} = \frac{1}{2A}\begin{Bmatrix} y_2 - y_3, & y_3 - y_1, & y_1 - y_2 \\ x_3 - x_2, & x_1 - x_3, & x_2 - x_1 \end{Bmatrix}$, with x_i, y_i the Cartesian coordinates of the nodes of the element

$[K] = \begin{bmatrix} k_{xx} & k_{xy} \\ k_{yx} & k_{yy} \end{bmatrix}$, with k_{xx}, k_{xy}, k_{yy} the components of the permeability tensor, K

$\{h^n\} = \begin{Bmatrix} h_1 \\ h_2 \\ h_3 \end{Bmatrix}$, with h_i the total head at the nodes of the element

$\{L\}^T = \{L_1, L_2, L_3\}$, with L_i the area coordinates

$\lambda = \rho_w g m_2^w$

q = flow across the perimeter of the element

A = area of the element

s = perimeter of the element

t = time

For transient flow, equation 3 is integrated in time using the time centered, Crank-Nicholson finite difference scheme (Zienkiewicz, 1977). The relationship between nodal heads in two successive time steps can be expressed as follows:

$$\left[\frac{2}{\Delta t}[C] + [D]^{t+\Delta t/2}\right]\{h^n\}_{t+1} = \left[\frac{2}{\Delta t}[C] - [D]^{t+\Delta t/2}\right]\{h^n\}_t + 2\{F\}^{t+\Delta t/2} \quad (4)$$

where: $[D]^{t+\Delta t/2} = \{B\}^T [K]^{t+\Delta t/2} \{B\} A$, with D being the stiffness matrix evaluated at $t+\Delta t/2$

$$[C] = \frac{A}{12}\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$, with C being the capacitance matrix

$\{F\}^{t+\Delta t/2}$ = load vector reflecting the boundary conditions evaluated at $t+\Delta t/2$

$\{h^n\}_t$ = known nodal total heads at time, t

$\{h^n\}_{t+1}$ = unknown nodal total heads at time, t + 1

COMPUTER IMPLEMENTATION

The formulated finite element solutions are computer implemented in the programs SEEP and TRASEE. The computer facility used is the DIGITAL VAX 11/780 system at the College of Engineering, University of Saskatchewan, Saskatoon.

TRASEE is a two-dimensional finite element computer program that can model both steady state and transient flow conditions. TRASEE can accommodate complex geometries with arbitrary degrees of heterogeneity and anisotropy for up to 12 different soils. The present version of TRASEE has the capability of modelling a discretized soil system of up to 1300 elements and 800 nodes. Anisotropic conditions can also be handled where the direction of the major coefficient of permeability is at any specified angle to the x-axis.

Boundary conditions of either total head or flow can be specified at certain nodes. Flow boundary conditions such as infiltration and evapotranspiration and pumping can be simulated by designating a positive or negative flow at certain nodes. Changing boundary conditions during the transient process can be handled as a prescribed step function. A special procedure is also included for revising the boundary conditions along seepage faces.

Three types of functions are used to describe the relationship between the coefficient of permeability and the pressure head. The constants of each function type can be specified to fit the actual behaviour when experimental data are available.

The computer programs calculate nodal heads, element velocities and gradients and average nodal velocities and gradients. The program include some capabilities for plotting the problem geometry and various nodal data. The computer program

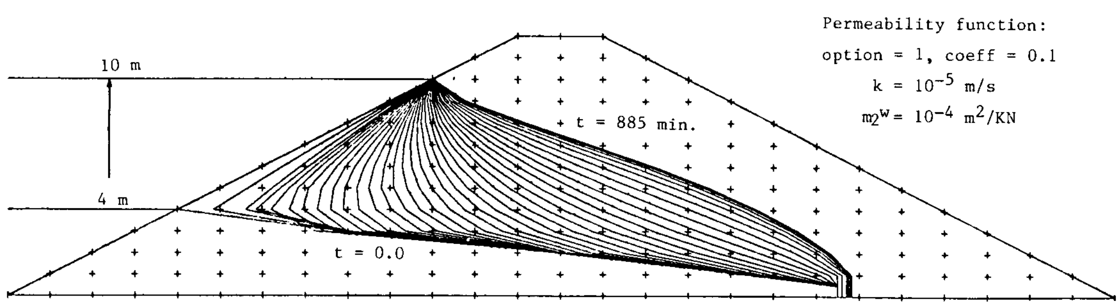

Fig. 1 Homogenous dam with a horizontal drain

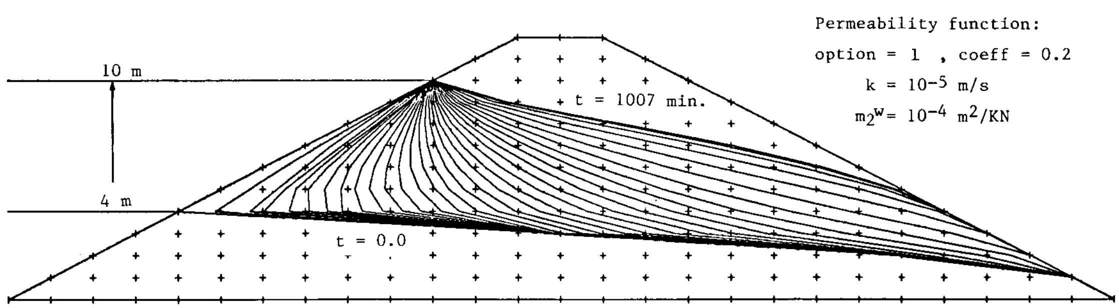

Fig. 2 Homogenous dam with an impervious lower boundary

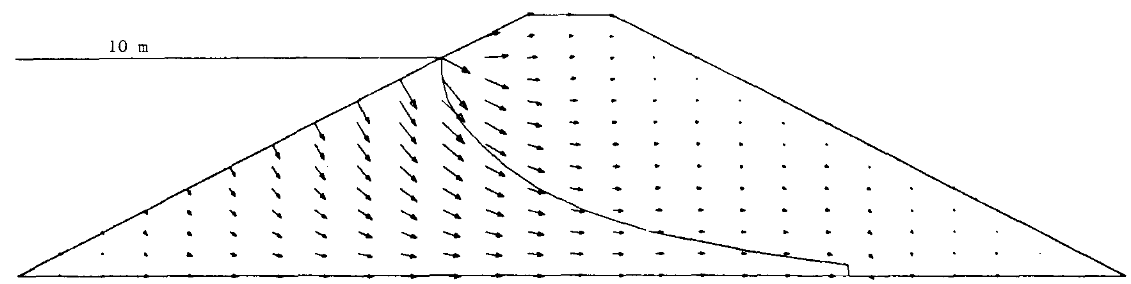

Fig. 3 Velocity field at t = 32 minutes

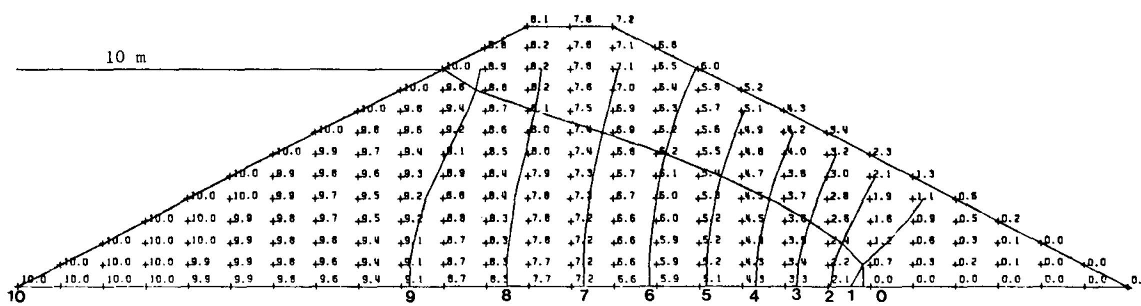

Fig. 4 Total nodal heads at final steady-state

SEEP and TRASEE are fully documented by Papagianakis (1982) and Lam (1983), respectively.

RESULTS

Two example problems are presented. Example 1 is a homogeneous dam with a horizontal drain, and example 2 is a homogeneous dam with an impervious lower boundary. In both cases, the reservoir level is raised from 4 meters to 10 meters instantaneously at time equal to zero. Figures 1 and 2 illustrate the rising of the phreatic lines with time for examples 1 and 2, respectively. For example 1, Fig. 3 illustrates the velocity field 32 minutes after the raising of the reservoir level, and Fig. 4 illustrates the total nodal heads for the final steady-state condition.

DISCUSSION AND CONCLUSION

Figures 1 and 2 illustrate the change in phreatic line as it advances from the initial condition to the final steady state condition. The phreatic line advances rapidly during the early stages of the transient process, and the change slow down as it approaches the steady state condition.

The fact that the phreatic line is not a flow line can be confirmed by the transient solution presented in Fig. 3 . In this figure, the average nodal velocity vectors indicate that there is a considerable amount of water flow across the phreatic line.

The results have been compared with Casagrande's classic solutions based on the flow net technique and other seepage analysis models. The solution of the final steady state condition (Fig. 4) compares satisfactory with the flow net solution. The small differences are due to the fact that the model takes into account water flow in the unsaturated zone, while it is not considered in the flow net technique. Another feature is that the equipotential lines extend all the way through the unsaturated zone. This means that there is potential gradient in the unsaturated zone, and there can be water flow in the unsaturated zone. This agrees with the solution by Freeze (1971).

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