A mathematical model for predicting moisture flow in an unsaturated soil under hydraulic and temperature gradients

V. Dakshanamurthy and D.G. Fredlund

Abstract: A theoretical model is presented to predict the moisture flow in an unsaturated soil as the result of hydraulic and temperature gradients. A partial differential heat flow equation (for above-freezing conditions) and the two partial differential transient flow equations (one for the water phase and the other for the air phase), are derived in this paper and solved using a finite difference technique. Darcy's law is used to describe the flow in the water phase, while Fick's law is used for the air phase. The constitutive equations proposed by Fredlund and Morgenstern are used to define the volume change of an unsaturated soil. The simultaneous solution of the partial differential equations gives the temperature, the pore-water pressure, and the pore-air pressure distribution with space and time in an unsaturated soil. The pressure changes can, in turn, be used to compute the quantity of moisture flow.

Key words: moisture flow, thermal gradients, hydraulic gradients, vapor flow, matric suction, unsaturated soil.

Introduction

Increasing interest has been shown in recent years in understanding the behavior of unsaturated soils, especially those exhibiting high swelling or shrinking characteristics. The prediction of moisture flow under transient conditions is important in both soil science and civil engineering. Geotechnical engineers are interested in moisture flow when considering such practical problems as the design of shallow foundations, highway and airfield pavements, and the stability of unsaturated soil slopes. The prediction of changes in moisture content for any given climate and topographic location involves many complex variables such as varying soil properties, the type of ground cover and surrounding vegetation, groundwater table depth, wind velocity, solar radiation, and atmospheric conditions. Because of the complex nature of the problem, research work in this area has largely remained semi-empirical with serious approximations.

The theory by Terzaghi (1943) of one-dimensional consolidation for saturated soils has formed an extremely useful transient flow formulation for geotechnical engineering. However, there has been difficulty in extending the transient flow solutions to embrace unsaturated soils. In general, proposed analyses for transient flow have assumed that the volume change of the soil is inconsequential. Most transient flow analyses presently used in geotechnical engineering have been borrowed from the soil science literature without careful consideration of the differences associated with the geotechnical engineering problems.

By the early 1950's there was a considerable interest in the physics of moisture movement in porous media, under both isothermal and non-isothermal conditions. The complex nature of the pore space in soil and the water held therein made it difficult to understand the force fields acting on the water. Philip and de Vries (1957) presented the following equation, which describes moisture and heat transfer under combined moisture content and temperature gradients.

$$[1] \quad Q = D_\theta \nabla\theta + D_T \nabla T + K_\theta$$

where:

Q = net water flux,
D_θ = isothermal moisture diffusivity,
$\nabla\theta$ = moisture content gradient,
D_T = thermal diffusivity,
∇T = temperature gradient, and
K_θ = gravity term.

The terms D_θ and D_T are each made up of two components: one for vapor flow and the other for liquid flow.

Taylor and Cary (1964) proposed linear flow equations based on the law of irreversible thermodynamics. The formulation considers the flux of one component of a system to influence the flux of other components of the system. For example, the calculation of the heat flux is affected by the calculation of the water flux.

Cassel et al. (1969) reviewed the theoretical models proposed for predicting the movement of water in soils in response to an imposed temperature gradient and concluded that the work by Philip and de Vries (1957) showed acceptable agreement of the predicted and ob-

V. Dakshanamurthy. Graduate Student, Department of Civil Engineering, University of Saskatchewan, 57 Campus Drive, Saskatoon, SK, Canada S7N 5A9.
D.G. Fredlund. Professor, Department of Civil Engineering, University of Saskatchewan, 57 Campus Drive, Saskatoon, SK, Canada S7N 5A9.

Reproduced with permission from the *Water Resources Research*, 17(3): 714–722, 1981.

served water movement. It was concluded that the Taylor-Cary equation (Taylor and Cary 1964) underpredicted the observed water movement.

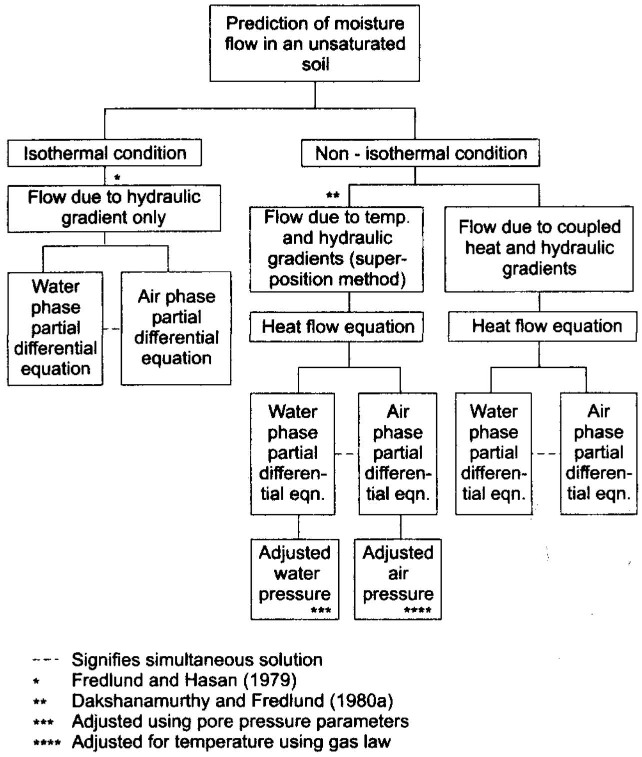
Nachlinger and Lytton (1969) proposed two coupled differential equations representing an isothermal case and three coupled differential equations representing the non-isothermal case for flow through a porous medium. The formulation was based on the principle of superimposed continuum mechanics. The driving forces associated with the liquid and gas phases were the hydraulic and thermal gradients.

Researchers in soil sciences (Cassel et al. 1969) and geotechnical engineering (Aitchison et al. 1965; Dempsey and Elzeftawy 1976; Dempsey 1978; Sophocleous 1978) have appeared to generally accept the theory of Philip and de Vries (1957). However, the adoption of the Philip-de Vries model in geotechnical engineering practice has some limitations. For example, the assumption that the soil is incompressible is not realistic. This creates difficulty in analysing geotechnical problems such as the dissipation of pore pressures in the compacted core material of an earthfill dam. Here the total stress is increasing with time (during placement of fill) and the pore-water and pore-air pressures induced are a function of the applied load and the compressibility of the soil. In order to solve this problem, the soil must be assumed to be compressible and the constitutive relations for the soil must be known in terms of total stress and pore-water and pore-air pressure. Generally, the flow of the air phase is ignored. However, fundamentally the air and water phases flow independently under total hydraulic heads. Consequently, it is not matric suction that produces flow in the water phase. In fact, it is possible to have the flow of water in one direction while the flow of air is in another direction.

Fredlund and Hasan (1979) presented a one-dimensional consolidation theory for unsaturated soils. Two partial differential equations were derived, one for the water phase and the other for the air phase. These were shown to describe the transient flow processes taking place as a result of the application of an external load to an unsaturated soil. Dakshanamurthy and Fredlund (1980*a*) further developed the model to predict the moisture flow in an unsaturated soil under isothermal and non-isothermal conditions. The model proposed was also extended to two-dimensional geometrical conditions. Two partial differential equations, one for the water phase and the other for the air phase, were again used to describe the moisture flow under transient conditions. In addition, a partial differential heat flow equation was solved and the corresponding pore-water water and pore-air pressures were adjusted by the method of superposition, in order to account for the imposed temperature gradient. The flow chart shown in Fig. 1 illustrates the various stages in the formulation of the moisture flow model.

This paper combines the effect of the thermal gradient and the hydraulic gradient to obtain a modified form of the pore-air and pore-water partial differential equations. In total, three partial differential equations, namely, (1) heat flow, (2) water phase, and (3) air phase equations, are solved simultaneously for changes in the combined thermal and hydraulic boundary conditions. Temperature changes in the soil are converted to a change in pore-air pressure. The changes in pore-air pressure produce changes in the pore-water pressure in accordance with the B_{aw} pore pressure parameter defined in geotechnical engineering (Fredlund 1976; Hasan and Fredlund 1980).

Fig. 1. Flow chart for prediction of moisture flow.



$$[2] \qquad B_{aw} = \frac{\nabla u_a}{\nabla u_w}$$

where:

Δu_a = change in pore-air pressure, and
Δu_w = change in pore-water pressure.

The B_{aw} pore pressure parameter is a function of the compressibility of the air-water mixture and the compressibility of the soil structure. B_{aw} is less than 1 for an unsaturated soil and approaches a value of 1 as saturation is approached.

In this study, attention is given to solving example problems typical of the conditions which would exist in an unsaturated soil layer below ground surface. The examples reflect changes that would occur in a subgrade either due to continuous evaporation or infiltration, coupled with increases or decreases in temperatures. It is assumed that the boundary conditions are known at all times and, in fact, remain constant during the process under consideration. Only one-dimensional heat and moisture flow is presently considered. The distribution of moisture flow enables the prediction of the overall volume change, the degree of saturation, and the void ratio with time.

The proposed model

All the necessary physical equations required for the formulation of a moisture flow model have been previously proposed and experimentally verified (Fredlund 1973; Fredlund and Morgenstern 1976, 1977). These are the continuity equation and the constitutive relations for the soil structure, the water phase, and the air phase. $(\sigma_y - u_a)$ and $(u_a - u_w)$ are used as the stress state variables for an unsaturated soil where σ_y is the total stress in the y-direction, u_a is the pore-air pressure, and u_w is the pore-water pressure. The flow equation for the water phase is described by Darcy's law (Childs and Collis-George 1950), and Fick's law is used to describe flow in the air phase (Blight 1971).

The one-dimensional transient flow equations for an unsaturated soil are derived in a manner so as to reflect the form of derivation used in geotechnical engineering for saturated soils. The conventional assumptions for the consolidation theory developed by Terzaghi (1943) are adhered to along with the following additions:

(1) The air phase is continuous. This is generally the case when the degree of saturation is less than approximately 85%. At higher degrees of saturation the air phase generally becomes occluded and the formulation would need to treat the soil as having one compressible fluid phase.
(2) The coefficients of permeability with respect to the water and air phases are functions of the volume-weight properties of the soil (Corey 1957). The coefficients are assumed to be constant in the derivation of the partial differential equations; however, they can be revised in accordance with any proposed function throughout the transient process.
(3) The volume change moduli remain constant during the transient process.
(4) Vapor pressure gradients and the dissolving of air in the water are not considered in the analysis.

The above assumptions are not completely accurate throughout the processes being considered. For example, the coefficient of permeability with respect to the water phase is a function of both water content and degree of saturation. However, the coefficient of permeability is assumed to be constant in the derivation of the transient flow equation. The derived equations are later solved using a finite difference technique. During the solution of the equations it is possible to update the coefficient of permeability in accordance with any desired functional relationship. For example, the coefficient of permeability could be written as a function of the degree of saturation (Corey 1957). The degree of saturation (as well as water content and void ratio) are known at all times since the constitutive relations for the unsaturated soil are known in terms of the stress state variables. The coefficient of permeability with respect to the air phase is also a function of water content and degree of saturation, but it is likewise assumed to be constant. The above approach can again be justified since the partial differential equations are being solved using a finite difference technique which allows for the adjustment of the coefficients of permeabilities with time. The constitutive relations for the unsaturated soil have non-linear soil moduli; however, they are assumed linear for the example problems presented. These values could also be adjusted during the finite difference solution, if desired.

The proposed model does not include the effect of vapor moisture movement as a result of a vapor pressure gradient. In other words, it is assumed that the vapor pressure above the boundary of the layer under consideration is the same as the vapor pressure within the soil. However, moisture vapor can be assumed to be carried in or out of the soil as a result of air movement. Further research presently underway at the University of Saskatchewan in Saskatoon considers a more rigorous analysis of vapor pressure gradients.

Water phase partial differential equation

Let us consider a differential soil element as shown in Fig. 2. In the transient flow process, water flows out of the element with time. The constitutive relationship for the water phase defines the volume of water in the element for any combination of total, water, and air pressures (Fredlund and Morgenstern 1976).

The volume of water entering and leaving the element can be described by Darcy's law as follows:

[3] $$V_{WE} = -\left[k\frac{\partial h_w}{\partial y}\right]\mathrm{d}x\,\mathrm{d}z$$

[4] $$V_{WL} = -\left[k\frac{\partial h_w}{\partial y} + \frac{\partial}{\partial y}\left(k\frac{\partial h_w}{\partial y}\right)\mathrm{d}y\right]\mathrm{d}x\,\mathrm{d}z$$

where:

V_{WE} = volume of water entering the element,
V_{WL} = volume of water leaving the element,
k = coefficient of permeability with respect to the water phase,
h_w = hydraulic head in the water phase, and
y = direction of flow.

The net flux of water per unit volume of the element is

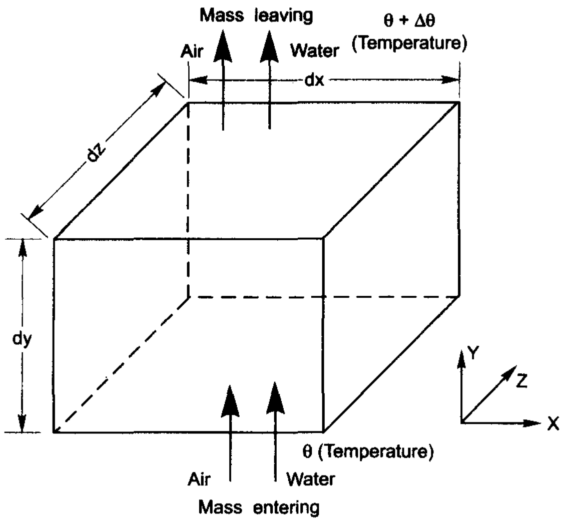
$$\frac{\partial(V_w / V)}{\partial t} = -\left[\frac{k}{\gamma_w}\frac{\partial^2 u_w}{\partial y^2}\right]$$

where:

γ_w = unit weight of water,
V = total volume of a referential element of the soil mass, and
t = time.

Equation [5] is equated to the time differential of the constitutive relationship for the water phase:

Fig. 2. A differential element in the soil mass, showing coupled heat, air, and water flow.



$$[6] \quad m_1^w \frac{\partial(\sigma_y - u_a)}{\partial t} + m_2^w \frac{\partial(u_a - u_w)}{\partial t} = -\left[\frac{k}{\gamma_w}\frac{\partial^2 u_w}{\partial y^2}\right]$$

where:

m_1^w = slope of $(\sigma_y - u_a)$ versus volume of water plot when $d(u_a - u_w)$ is zero, and
m_2^w = slope of $(u_a - u_w)$ versus volume of water plot when $d(\sigma_y - u_a)$ is zero.

The change in total stress with respect to time is set to zero for the type of example problems being considered. By simplifying and rearranging eq. [6], the water phase partial differential equation can be written:

$$[7] \quad \frac{\partial u_w}{\partial t} = C_w \frac{\partial u_a}{\partial t} + c_v^w \frac{\partial^2 u_w}{\partial y^2}$$

where:

$C_w = -(1 - m_2^w / m_1^w) / (m_2^w / m_1^w)$, interactive constant associated with the water phase equation.

This equation is further simplified by letting $R_w = m_2^w / m_1^w$. As the soil approaches saturation, R_w approaches unity.

$$[8] \quad c_v^w = \frac{1}{R_w}\frac{k}{\gamma_w}\frac{1}{m_1^w}$$

where:

c_v^w = coefficient of consolidation for the water phase.

Air phase partial differential equation

The air phase is compressible and flow occurs in response to a pressure gradient. The constitutive relationship for the air phase defines the volume of air in the element for any combination of the total water, and air pressures. Quantitatively, the air phase constitutive relationship is equal to the difference between the soil structure constitutive relationship and the water phase constitutive relationship.

According to Fick's law, the masses of air entering and leaving the element are:

$$[9] \quad M_{AE} = -D \frac{\partial p}{\partial y} dx\, dz$$

$$[10] \quad M_{AL} = -\left[D\frac{\partial p}{\partial y} + \frac{\partial}{\partial y}\left(D\frac{\partial p}{\partial y}\right)dy\right] dx\, dz$$

where:

M_{AE} = mass of air entering the element,
M_{AL} = mass of air leaving the element,
D = transmission constant, equal to $D^*(\omega / R\theta)$, having the same units as coefficient of permeability,
D^* = mass diffusivity of gases in air,
p = absolute air pressure (i.e., $u_a + u_{atm}$), and
u_{atm} = atmospheric air pressure.

The net flux of air through the element is,

$$[11] \quad \frac{\partial m}{\partial t} = -\left(D\frac{\partial^2 p}{\partial y^2}\right)$$

where:

m = the mass of air in the element.

The mass rate of change is written in terms of a volume rate of change by differentiating the relationship between the mass and volume of air.

$$[12] \quad \frac{\partial(V_a / V)}{\partial t} = \frac{\partial(m / \gamma_a)}{\partial t} = \frac{1}{\gamma_a}\frac{\partial m}{\partial t} - \frac{1}{\gamma_a^2}\frac{\partial \gamma_a}{\partial t} m$$

The density of air can be written in terms of pressure and temperature using the gas law.

$$[13] \quad \gamma_a = \frac{\omega}{R\theta} p$$

where:

ω = molecular weight of air,
R = universal gas constant, and
θ = absolute temperature.

The mass of air is written in terms of the density of air, γ_a, the degree of saturation, S, and the porosity of the soil, n.

$$[14] \quad m = (1 - S) n \gamma_a$$

By differentiating eq. [13] and substituting eqs. [11] and [14] and by simplifying and rearranging, eq. [12] can be written as:

$$[15] \quad \frac{\partial(V_a / V)}{\partial t} = -\frac{D}{\gamma_a}\frac{\partial^2 p}{\partial y^2} - \frac{(1-S)n}{p}\frac{\partial p}{\partial t} + \frac{(1-S)n}{\theta}\frac{\partial \theta}{\partial t}$$

Equation (15) can be equated to the time differential of the air phase constitutive relationship:

$$[16]\quad m_1^a \frac{\partial(\sigma_y - u_a)}{\partial t} + m_2^a \frac{\partial(u_a - u_w)}{\partial t} = -\left[\frac{DR\theta}{\omega p}\frac{\partial^2 u_a}{\partial y^2} + \frac{(1-S)n}{p}\frac{\partial p}{\partial t} - \frac{(1-S)n}{\theta}\frac{\partial \theta}{\partial t}\right]$$

where:

m_1^a = compressibility moduli of the air phase associated with the $(\sigma_y - u_a)$ stress state variable, and

m_2^a = compressibility moduli of the air phase associated with the $(u_a - u_w)$ stress state variable.

As in the water phase equation the change in total stress with respect to time is set to zero. By simplifying and rearranging eq. [16] the air phase partial differential equation can be written as:

$$[17]\quad \frac{\partial u_a}{\partial t} = C_a \frac{\partial u_w}{\partial t} + C_\theta \frac{\partial \theta}{\partial t} + c_v^a \frac{\partial^2 u_a}{\partial y^2}$$

where:

$$[18]\quad C_a = \frac{-m_2^a / m_1^a}{(1 - m_2^a / m_1^a) + \{[(1-S)n] / [m_1^a(u_a + u_{atm})]\}}$$

(which is the interactive pressure constant associated with the air phase equation). This equation is further simplified by letting $R_a = m_2^a / m_1^a$. When the soil is saturated, R_a is equal to unity.

$$[19]\quad C_\theta = \frac{1}{\theta}\left[\frac{(1-S)n(u_a + u_{atm})}{(1-R_a)(u_a + u_{atm})m_1^a + (1-S)n}\right]$$

(which is the interactive thermal constant associated with the air phase equation).

$$[20]\quad c_a^v = \frac{DR\theta}{\omega}\left[\frac{1}{(1-R_a)(u_a + u_{atm})m_1^a + (1-S)n}\right]$$

(which is the coefficient of consolidation for the air phase).

The dissipation of the excess pressure of the pore-air and pore-water phases are obtained by solving eqs. [7] and [17] simultaneously, using the explicit finite difference method described in the Appendix. However, prior to solving eqs. [7] and [17] it is necessary to obtain the thermal gradient within the soil.

Heat flow equation

The heat flow equation for one-dimensional conditions is as follows (Aldrich 1956):

$$[21]\quad \frac{\partial \theta}{\partial t} = \alpha \frac{\partial^2 \theta}{\partial y^2}$$

where:

θ = temperature,

λ = thermal conductivity,

c = heat capacity,

α = thermal diffusivity factor, equal to $(\lambda / c\gamma)$, and

γ = soil density.

The heat flow equation is solved using an explicit finite difference method (see Appendix). The solution gives the change in temperature with one-dimensional space and time (Fig. 3).

Example problems

Particular emphasis is given to solve four example problems involving the prediction of moisture flow in an unsaturated subgrade soil such as that below a highway or airfield pavement. The model simulates the effect of environmental change on a layer of soil within the subgrade. It is assumed that the subgrade system is initially in a state of equilibrium. Then a sudden environmental change is imposed at the boundary. Changes considered are: evaporation, infiltration, and temperature changes. The environmental changes either due to evaporation or infiltration builds up an excess (either positive or negative with respect to the equilibrium state) pore-water pressure at the boundary. Similarly, the maximum or minimum temperature change also results in an excess pore-air pressure. It is assumed, under non-isothermal conditions, that the change in excess pore-water pressure consequent to the change in excess pore-air pressure is zero. In other words, the pore pressure parameter, B_{aw}, relating pore-air pressure and pore-water pressure, is assumed to be large. Other, more realistic, lower B_{aw} values could also have been assumed. These excess pore-water and pore-air pressures build up at the surface and cause simultaneous flow in the water and the air phases independently. An unsaturated soil mass will equilibrate to a new set of pore pressure conditions dependent on the assumed final boundary conditions. The boundary conditions used in the four example problems discussed below are presented in Table 1. The solution to the transient flow conditions is obtained by solving simultaneously the three partial differential equations, namely, eqs. [7], [17], and [21]. The details of the simultaneous solutions of these equations are given in the Appendix.

A computer program has been developed to solve these partial differential equations (Dakshanamurthy and Fredlund 1980*b*). Solutions are presented for various initial and final pore-water and pore-air pressures boundary conditions and various initial and final surface temperatures. The effect of various compressibility moduli for the water and air phase constitutive relations is also investigated along with the effect of degree of saturation, porosity, and thermal conductivity and heat capacity.

In the example problems solved, the soil properties used are typical of Regina clay found near Regina, SK, Canada (Fredlund 1964) (Table 2). For this study a 10 cm

Table 1. Boundary conditions for example problems.

Example number	Corresponding figure number	Pore-water pressure, kPa		Pore-air pressure, kPa		Temperature, °C		Remarks
		Initial	Final	Initial	Final	Initial	Final	
1	4, 6	–280	–420	102	102	20	20	isothermal, consolidation
2	3, 4, 6	–280	–420	102	102	10	25	non-isothermal, consolidation
3	5, 7	–420	–280	102	102	20	20	isothermal, swelling
4	3, 5, 7	–420	–280	102	102	10	25	non-isothermal, swelling

Table 2. Summary of classification tests, compressibility, and permeability properties of Regina clay.

	Value
Specific Gravity	2.83
Atterberg limits	
Liquid limit	75.5%
Plastic limit	24.9%
Shrinkage limit	13.1%
Plasticity index	50.6
Grain size distribution	
Sand sizes	8%
Silt sizes	41%
Clay sizes	51%
Compressibility*	
m_1^w	0.0007614/kPa
m_1^a	0.0003263/kPa
$R_w = R_a$	0.7
Permeability*	
k	0.6×10^{-10} m/s
D^*	1.0×10^{-9} m²/s
Volume-weight properties	
Soil density (γ)	1.8g/cm³
Porosity (n)	50%
Degree of saturation (S)	70%
Thermal properties	
λ = thermal conductivity	0.4574 cal/m s °C
c = heat capacity	30 cal/g K
Physico-chemical properties	
ω = molecular weight of air	18.015 g/mol
R = universal gas constant	847.825 J/mol K

*Estimated values.

thick layer of compacted Regina clay is used in the example problems.

Results and discussions

The distribution of temperature, pore-water pressure, and pore-air pressure are obtained by solving eqs. [21], [7], and [17], respectively. The water content (volumetric) is obtained by back substitution into the water phase constitutive relationship. Because of the assumption that the change in excess pore-water pressure is zero under non-isothermal conditions, the water content (volumetric) distribution with space and time is essentially the same for isothermal and non-isothermal conditions. The results are presented by the family of curves shown in Figs. 3 to 8.

Figure 3 shows the temperature isotherms within the clay layer as a result of an increase in the temperature from 10°C (283.2 K) to 25°C (298.2 K). The figure illustrates how the imposed thermal gradient at the surface slowly dissipates to the bottom of soil and eventually equilibrates to the new boundary condition.

Figure 4 shows the pore-water pressure distribution throughout the clay layer due to a change in the pore-water pressure at the boundary. The initial (i.e., equilibrium) pore-water pressure was –280 kPa, and the boundary pore-water pressure was changed to a value of –420 kPa at the surface instantaneously. The change in pore-water pressure is assumed to be due to evaporation. However, this is a special case of evaporation where the boundary pore-water pressure is kept constant. Figure 5 shows the numerically identical but reverse process to that presented in Figure 4. In other words, the surface of the soil is kept moistened to maintain the pore-water pressure at –280 kPa.

Figures 6 and 7 show the pore-air pressure distributions throughout the clay layer due to changes in the pore-water pressure under isothermal and non-isothermal conditions. Figure 6 shows the distribution under the volume decrease or consolidation process, and Fig. 7 shows the distribution under the volume increase or swelling process.

Figures 4 to 6 show that it takes considerable time (i.e., 854 hours) to dissipate the excess pore-water and pore-air pressure to the bottom of 10 cm thick clay layer. This is understandable, since the permeability and compressibility moduli are low for the soil considered. However, the system reaches a new equilibrium condition, and the example demonstrates the interaction between the effects of a thermal and hydraulic gradient.

Figure 8 shows the distribution of water content (volumetric) throughout the clay layer considered, under both the consolidation and the swelling process. The magnitude of the changes in water content are equal for the swelling and consolidation process because the compressibility moduli have been chosen as equal. In other words, hysteretic effects are not considered for this example problem. The final water content throughout the clay layer equilibrates to a value consistent with the change in pore-water pressure at the surface. It is also possible to predict changes in overall volume of the soil or heave of the soil surface. The distribution of moisture

Fig. 3. Temperature isotherms for an example problem.

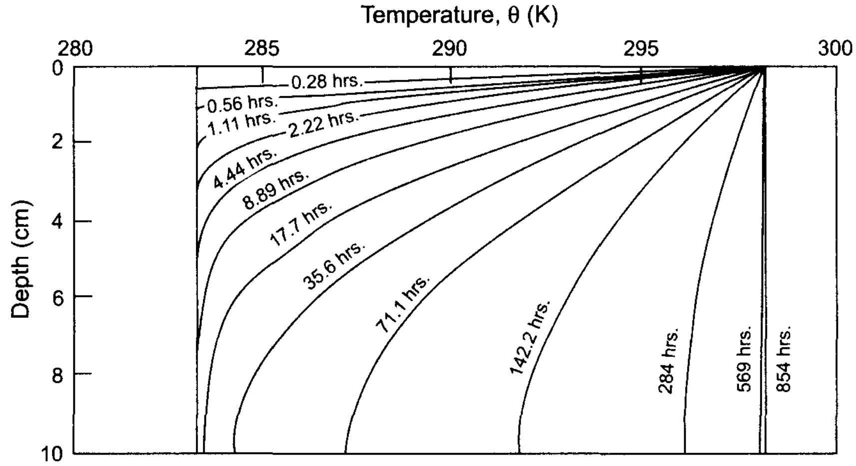


Fig. 4. Pore-water pressure distribution under consolidation.

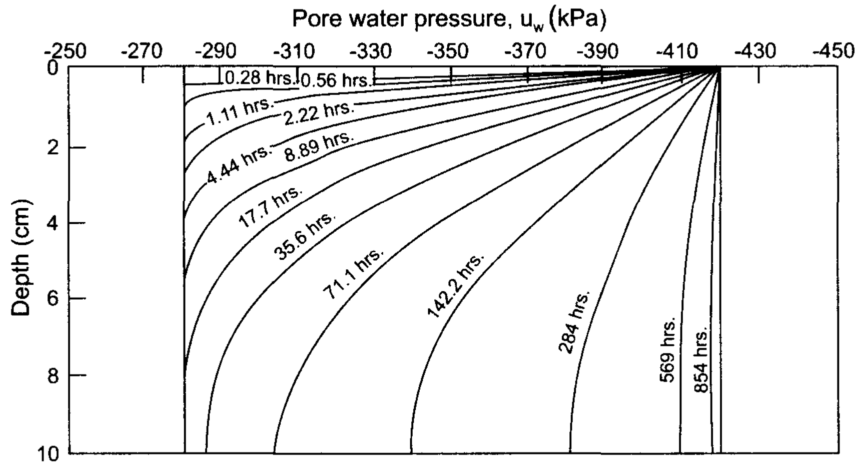
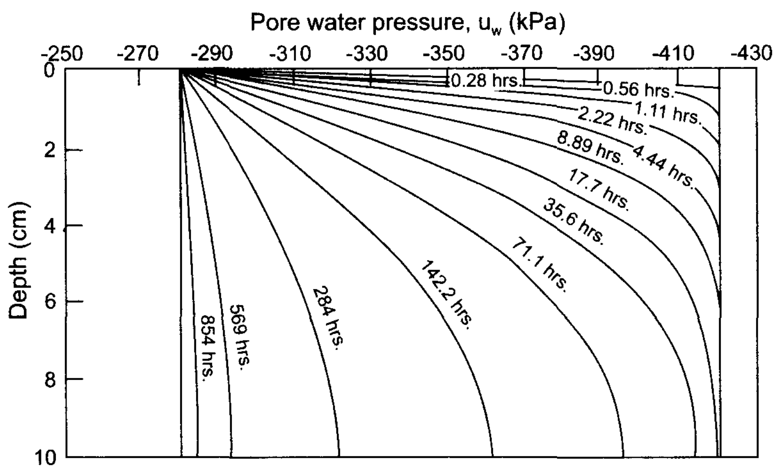


Fig. 5. Pore-water pressure distribution under swelling.



content with space and time and the changes in overall volume of the saturated soil system under varying boundary conditions enables the prediction of the degree of saturation, S, and the void ratio, e, during the transient processes.

The boundary conditions can be changed at any time during one process and the effects will be superimposed on one another. This results in highly complex thermal and pore pressure distributions throughout the soil layer.

Fig. 6. Pore-air pressure distribution under consolidation.

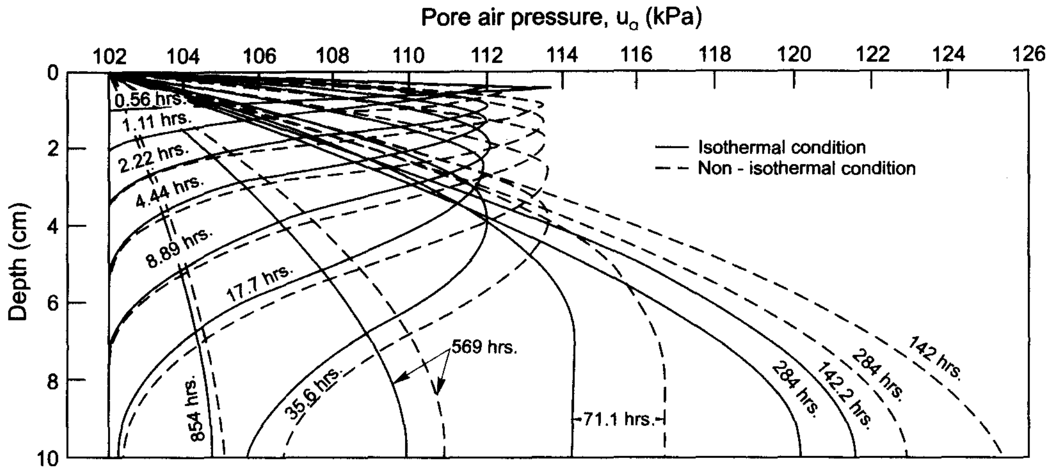


Fig. 7. Pore-air pressure distribution under swelling.

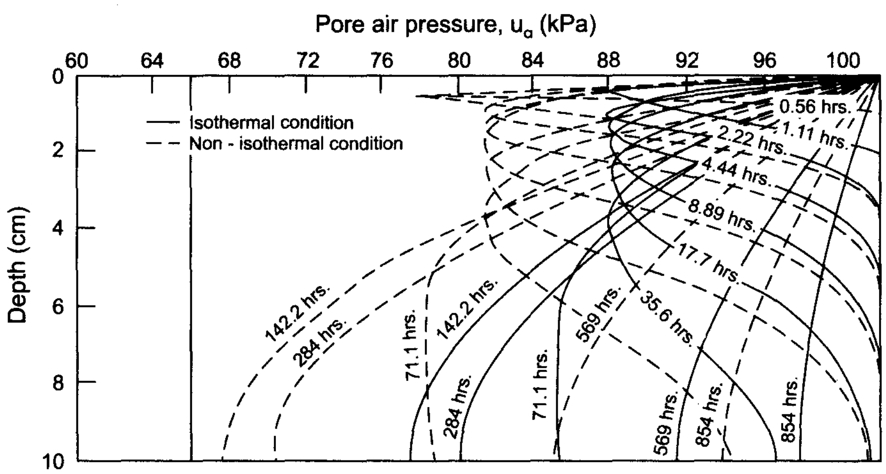
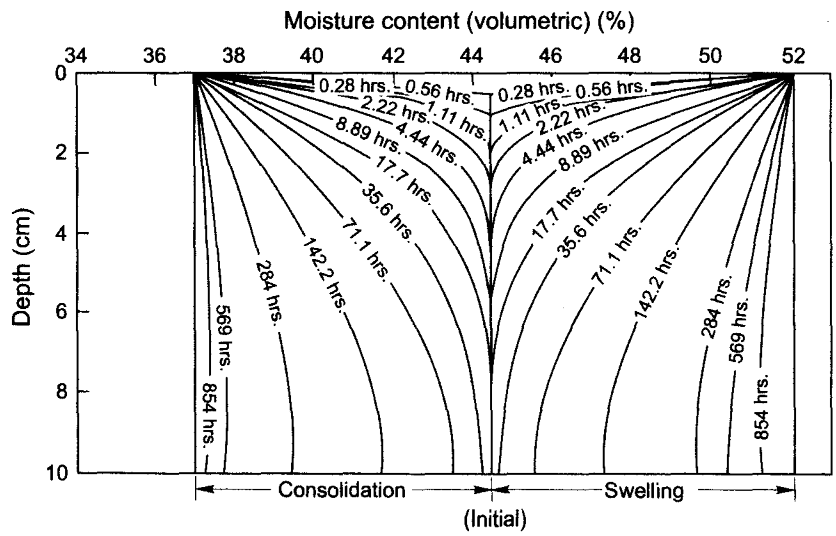


Fig. 8. Water content (volumetric) distribution.



Summary

All the necessary physical relations are available to formulate a rigorous theoretical model to describe the transient flow process under coupled hydraulic and thermal gradients in an unsaturated soil. The heat flow equation is first solved by a forward finite difference technique. Then two partial differential equations (i.e., one for the water phase and the other for the air phase) are solved simultaneously using a special finite difference procedure. Families of curves show temperature, pore-water pressure, pore-air pressure, and water content (volumetric) distribution throughout the clay layer considered, as a result of coupled hydraulic and temperature gradients in an unsaturated soil. The purpose of the example problems is to demonstrate the form of the resulting processes under various changes in the boundary conditions. Present analyses being performed at the University of Saskatchewan, Saskatoon, involve the refining of the coefficients of permeability during the transient processes.

The distribution of water content and overall volume change enables the prediction of overall volume-weight properties of the soil throughout the transient process. The model shows good promise for describing the behavior of unsaturated soil systems under highly complex environmental changes.

Appendix

Figure A1 shows the finite difference designations for the example problems. Prior to solving the water phase and air phase equations the heat flow eq. [21] is solved. The finite difference form of the heat flow equation is as follows:

[A1] $$\frac{(\theta_{i,j+1}-\theta_{i,j})}{\Delta t}=\alpha\frac{(\theta_{(i+1,j)}-2\theta_{(i,j)}+\theta_{(i-1,j)})}{\Delta y^2}$$

where:

i = array used for depth increments, and
j = array used for time increments.

[A2] $$\theta_{(i,j+1)}=\theta_{(i,j)}+\beta_t(\theta_{(i+1,j)}-2\theta_{(i,j)}+\theta_{(i-1,j)})$$

where:

$$\beta_t=\alpha\frac{\Delta t}{\Delta y^2}$$

A special numerical procedure is adapted to obtain the simultaneous solution of the water phase and air phase partial differential equations (i.e., eqs. [7] and [17]. These two equations are solved using an explicit forward difference technique. Since the air phase and water phase equations formulated in this paper are non-linear, this procedure is advantageous because this procedure transforms the non-linear partial differential equation into a linear partial differential equation. The following steps are used to solve the water phase and air phase transient flow equations.

Write the transient flow equation for the water phase in a finite difference form:

[A3] $$\frac{u_{w_{(i,j+1)}}-u_{w_{(i,j)}}}{\Delta t}=C_w\left[\frac{u_{a_{(i,j+1)}}-u_{a_{(i,j)}}}{\Delta t}\right]+c_v^w\left[\frac{u_{w_{(i+1,j)}}-2u_{w_{(i,j)}}+u_{w_{(i-1,j)}}}{\Delta y^2}\right]$$

Write the transient flow equation for the air phase in a finite difference form:

[A4] $$\frac{u_{a_{(i,j+1)}}-u_{a_{(i,j)}}}{\Delta t}=C_a\left[\frac{u_{w_{(i,j+1)}}-u_{w_{(i,j)}}}{\Delta t}\right]+C_\theta\frac{(\theta_{(i,j+1)}-\theta_{(i,j)})}{\Delta t}+c_v^a\left[\frac{u_{a_{(i+1,j)}}-2u_{a_{(i,j)}}+u_{a_{(i-1,j)}}}{\Delta y^2}\right]$$

Equation [A4] is multiplied by C_w and eqs. [A3] and [A4] are solved simultaneously for pore-water pressure (u_w). The resulting eq. [A5] is simplified and rearranged such, that the unknown pore-water pressure at the given time step (i.e., j + 1-th) is on the left-hand side and all known variables at the previous time step (i.e., j-th) are on the right-hand side of the equation:

[A5] $$u_{w_{(i,j+1)}}=u_{w_{(i,j)}}+\frac{\beta_w g_1^w}{(1-C_aC_w)}+\left(\frac{C_w}{1-C_aC_w}\right)\beta_a f_1^a+\frac{C_wC_\theta}{(1-C_aC_w)}\theta_1$$

where:

$$\beta_w=c_v^w\left(\frac{\Delta t}{\Delta y^2}\right)\qquad \beta_a=c_v^a\left(\frac{\Delta t}{\Delta y^2}\right)$$

$$g_1^w=u_{w_{(i+1,j)}}-2u_{w_{(i,j)}}+u_{w_{(i-1,j)}}$$

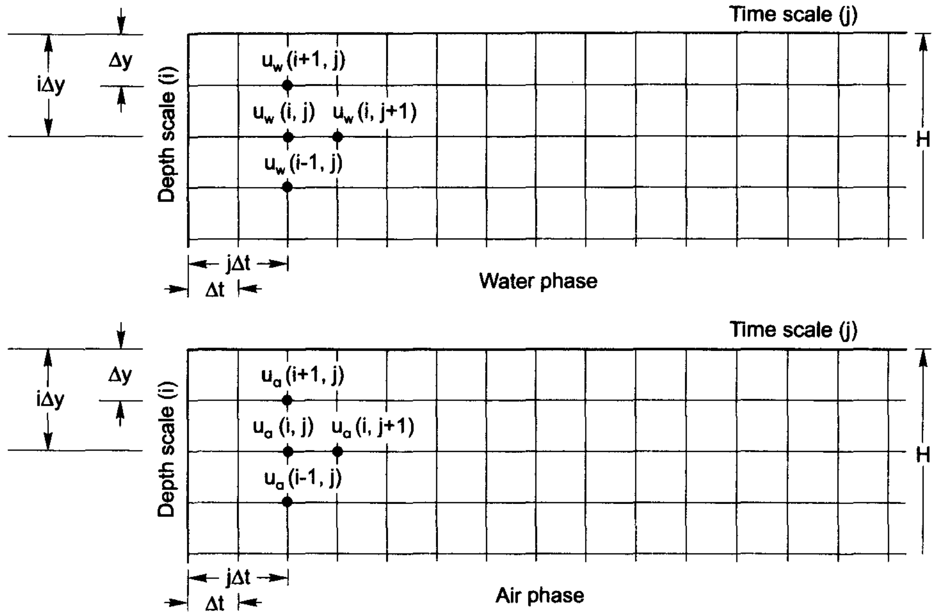
$$f_1^a=u_{a_{(i+1,j)}}-2u_{a_{(i,j)}}+u_{a_{(i-1,j)}}$$

$$\theta_1=[\theta_{(i,j+1)}/\theta_{i,j}-1]$$

Equation [A3] is multiplied by C_a and eqs. [A3] and [A4] are solved simultaneously for pore-water pressure (u_a). The resulting eq. [A6] is simplified and rearranged such, that unknown pore-air pressure at the given time step (i.e., j + 1-th) is on the left-hand side and all known variables at the previous time step (i.e., j-th) are in the right hand side of the equation:

[A6] $$u_{a_{(i,j+1)}}=u_{a_{(i,j)}}+\left(\frac{C_a}{1-C_aC_w}\right)\beta_w g_1^w+\frac{\beta_a f_1^a}{(1-C_aC_w)}+\frac{C_\theta}{(1-C_aC_w)}\theta_1$$

Fig. A.1. Finite difference mesh for the transient flow equations.



Compute the pore-water and the pore-air pressures at the given time step from the known values at the previous time step. When computations for both pore-water and pore-air pressures for all depth steps are completed, then march forward to the next time step. Repeat the above procedure up to the desired time interval. The solution of all the above example problems are obtained by setting the values of β_t, β_w and β_a, terms ≤ 0.5 in order to satisfy the stability conditions of the partial differential equations.

Acknowledgements

This study was supported under a research grant from the Saskatchewan Highways and Transportation Department, Government of Saskatchewan, Regina, SK. Acknowledgement is made to Dr. R. Manohar, Professor of Mathematics, University of Saskatchewan, for his assistance during this study.

References

Aitchison, G.D., Russam, K., and Richards, B.G. 1965. Engineering concepts of moisture equilibria and moisture changes in soils. *In* Moisture equilibria and moisture changes in soils beneath covered areas. A Symposium in Print. *Edited by* G.D. Aitchison, Butterworth, Sydney, Australia, pp. 7–21.

Aldrich, H.P., Jr. 1956. Frost penetration below highway and airfield pavements. Highway Research Board Bulletin, National Research Council, Washington, DC, **135**: pp. 124–149.

Blight, G.E. 1971. Flow of air through soils. ASCE, Journal of Soil Mechanics and Foundation Division, **97**(SM4): 607–624.

Cary, J.W. 1966. Soil moisture transport due to thermal gradients: Practical aspects. Soil Science Society of America Journal, **30**: 428–433.

Cassel, D.K., Nielsen, D.R., and Biggar, J.W. 1969. Soil-water movement in response to imposed temperature gradients. Soil Science Society of America Journal, **33**: 493–500.

Childs, E.C., and Collis-George, N. 1950. The permeability of porous material. Proceedings Royal Society of London, Series A, **201**: 392–405.

Corey, A.T. 1957. Measurement of water and air permeability in unsaturated soil. Soil Science Society of America Journal, **21**: 7–10.

Dakshanamurthy, V., and Fredlund D.G. 1980*a*. Moisture and air flow in an unsaturated soil. Proceedings, 4th International Conference on Expansive Soils, American Society of Civil Engineering, Denver, CO, Vol. 1, pp. 514–532.

Dakshanamurthy, V., and Fredlund, D.G. 1980*b*. Transient flow processes in unsaturated soils (temperature, relative humidity, evaporation and infiltration). CD-16.4, Transportation and Geotechnical Group, Department of Civil Engineering, University of Saskatchewan, Saskatoon, SK, Canada.

Dempsey, B.J. 1978. A mathematical model for predicting coupled heat and water movement in unsaturated soil. International Journal of Numerical Analytical Methods in Geomechanics, **2**: 19–36.

Dempsey, B.J., and Elzeftawy, A. 1976. Mathematical model for predicting moisture movement in pavement systems. Transportation Research Records, National Academy of Sciences, Washington, DC, **612**, pp. 48–55.

Fredlund, D.G. 1964. Comparison of soil suction and one-dimensional consolidation characteristics of a highly plastic clay. M.Sc. thesis, University of Alberta, Edmonton, AB, Canada.

Fredlund, D.G. 1973. Volume change behavior of unsaturated soils. Ph.D. thesis, University of Alberta, Edmonton, AB, Canada.

Fredlund, D.G. 1976. Density and compressibility characteristics of air-water mixtures. Canadian Geotechnical Journal, **13**(4), 386–396.

Fredlund, D.G., and Hasan, J.U. 1979. One-dimensional consolidation theory: Unsaturated soils. Canadian Geotechnical Journal, **16**(3), 521–531.

Fredlund, D.G., and Morgenstern, N.R. 1976. Constitutive relations for volume change in unsaturated soils. Canadian Geotechnical Journal, **13**(3), 261–276.

Fredlund, D.G., and Morgenstern, N.R. 1977. Stress state variables for unsaturated soils. ASCE, Journal Geotechnical Engineering Division, **103**(GT5): 447–466.

Hasan, J.U., and Fredlund, D.G. 1980. Pore pressure parameters for unsaturated soils. Canadian Geotechnical Journal, **17**(3), 395–404.

Nachlinger, R., and Lytton, R.L. 1969. Continuum theory of moisture movement and swell in expansive clays. Centre for Highways Research, University of Texas, Austin, TX, Research Report 118.2.

Philip, J.R., and de Vries, D.A. 1957. Moisture movement in porous materials under temperature gradients. Transactions American Geophysical Union, **38**(2):222–232.

Taylor, S.A., and Cary J.W. 1964. Linear equation for the simultaneous flow of matter and energy in a continuous soil system. Soil Science Society of America Journal, **28**, 167–172.

Terzaghi, K. 1943. Theoretical Soil Mechanics. John Wiley, New York.

Sophocleous, M.A. 1978. Analysis of heat and water transport in unsaturated-saturated porous media. Ph.D. thesis, University of Alberta, Edmonton, AB, Canada.