The Relationship between Limit Equilibrium Slope Stabilitv Methods

La Rélation des Méthodes de Limite d'Equilibre de Pente Stable

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SYNOPSIS. Some of the methods commonly used for analyzing slopes utilizing the principles of limit equilibrium are the Ordinary or Fellenius method, the simplified Bishop method, the Corps of Engineers method, the Janbu simplified and the Janbu generalized methods, the Spencer method, and the Morgenstern-Price method. The similarities and differences in these methods have been obscure, largely because of: 1) the lack of uniformity in formulating the equations of equilibrium, 2) the ambiguity concerning inter-slice forces and 3) the unknown limitations imposed by non-circular failure surfaces.

Theoretical studies have shown that a common formulation of the equilibrium equations can be used for all of the methods. The factor of safety equations have been derived with respect to moment equilibrium and with respect to force equilibrium, and all methods use either or both of these equations. In addition, the use of an inter-slice force, functional relationship (i.e., lambda, λ) has been used to relate and compare the factors of safety computed for the various methods. This approach has provided a clear and concise assessment of the variation in factors of safety that are computed by the different methods for typical cross-sections, pore-water pressures and soil profiles.

INTRODUCTION

During the past three decades, numerous methods have been proposed for performing the two-dimensional limit equilibrium method of slices (Wright, 1969). The methods most commonly used are:

i) The Ordinary method. Other names given to this method are the Fellenius, Swedish Circle and Conventional method.
ii) The simplified Bishop method.
iii) The Spencer method.
iv) The Janbu simplified and the Janbu generalized methods.
v) Force equilibrium methods such as the Lowe and Karafiath method, the Corps of Engineers method, and the Taylor modified Swedish method.
vi) The Morgenstern-Price method.

The similarities and differences in these methods have been obscure, largely because of the lack of uniformity in formulating the factor of safety equations, the ambiguity concerning interslice forces and the unknown limitations imposed by non-circular failure surfaces. Some attempts have been made to assess the quantitative differences in factors of safety obtained from the various methods (Bishop, 1955; Wright, 1969; Duncan and Wright, 1980). In general, the quantitative differences in factors of safety obtained by the various methods, are not substantial with the exception of the Ordinary method which can differ by more than 60 percent from the other methods (Whitman and Bailey, 1967). Some attempts have also been made to show the relationship between the various methods from a theoretical standpoint (Wright, 1969; Fredlund and Krahn, 1977; Naderi, 1977; Popescu, 1978; Janbu, 1980).

The object of this paper is to present a general limit equilibrium method of slices formulation in two-dimensions and to show how each of the methods listed above is a special case of the general formulation. (Hereafter, the general limit equilibrium method of slices is simply referred to as the GLE method). The Ordinary method becomes an exception which cannot be related to the general formulation since it does not satisfy Newtonian force principles at the interslices (Fredlund and Krahn, 1977). The paper also shows how the various methods of slices can be extended to non-circular slip surfaces. Example problems are used to demonstrate the relationship between the factors of safety.

GENERAL LIMIT EQUILIBRIUM METHOD OF SLICES (GLE METHOD)

The elements of statics that can be used to derive the factor of safety are the summation of forces in two directions and the summation of moments about a chosen point of rotation. These elements of statics, along with the failure criteria, are insufficient to make the slope stability problem determinate. Either additional elements of physics or an assumption regarding the direction or magnitude of some of the forces is required to render the problem determinate. All methods considered in this paper use the latter procedure and make an assumption concerning the interslice forces.

Theoretical studies have shown that factor of safety equations can be independently derived to satisfy moment equilibrium and force equilibrium of the slices contained above an assumed slip surface (Fredlund and Krahn, 1977). In addition, an assumed functional relationship is used to specify the direction of the interslice forces. Later in this paper, the various limit equilibrium methods of slices are viewed as special cases of the GLE formulation.

Circular Slip Surface

Figure 1 shows the forces involved in the derivation of moment and force equilibrium factor of safety equations for a circular slip surface.

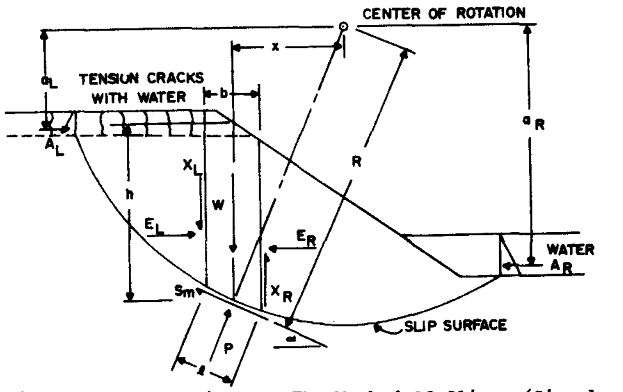

Fig. 1 Forces Acting For The Method Of Slices (Circular Slip Surface)

The definition of each variable is as follows:

- W = the total vertical force due to the mass of a slice of width 'b' and height 'h'.
- P = the total normal force on the base of a slice.
- S_m = the shear force mobilized on the base of each slice.
- E = the horizontal interslice normal forces.
- X = the vertical interslice shear forces.
- R = the radius or the moment arm associated with the mobilized shear force, S_m.
- x = the horizontal distance from the centroid of each slice to the center of rotation.
- a = the perpendicular distance from the resultant external water force to the center of rotation.
- b = the width of a slice.
- A = the resultant external water forces.
- α = the angle between the tangent to the center of the base of each slice and the horizontal.

The 'L' and 'R' subscripts on the 'E', 'X' 'a' and 'A' variables designate the left and right sides, respectively.

The magnitude of the shear force mobilized at the base of a slice can be written in terms of the Mohr-Coulomb failure criterion.

$$S_m = \frac{\ell}{F}\left[c' + (\sigma_n - u)\tan\phi'\right] \qquad [1]$$

where:

- c' = effective cohesion intercept
- ϕ' = effective angle of internal friction.
- σ_n = P/ℓ
- ℓ = length of the failure surface at the base of each slice.
- F = factor of safety.

A moment equilibrium equation for the GLE method is described for all slices by summing moments about the center of rotation.

$$\Sigma Wx - \Sigma S_m R \pm Aa = 0 \qquad [2]$$

The interslice shear and normal forces (i.e., X and E) do not appear directly in equation [2] since their summation over the overall slope must cancel.

The mobilized shear force, S_m, is written in terms of the shear strength criterion [1], and equation [2] can be solved for the factor of safety with respect to moment equilibrium, F_m.

$$F_m = \frac{\Sigma[c'\ell + (P - u\ell)\tan\phi']R}{\Sigma Wx \pm Aa} \qquad [3]$$

The force equilibrium equation for the GLE method is written by summing forces in the horizontal direction for the overall slope.

$$\Sigma P \sin\alpha - \Sigma S_m \cos\alpha \pm A = 0 \qquad [4]$$

Once again the interslice forces must cancel. The mobilized shear force is again written in terms of the failure criterion [1], and equation [4] can be solved for the force equilibrium factor of safety, F_f.

$$F_f = \frac{\Sigma[c'\ell + (P - u\ell)\tan\phi']\cos\alpha}{\Sigma P\sin\alpha \pm A} \qquad [5]$$

The normal force, P, for equations [3] and [5] can be evaluated by summing forces vertically on each slice.

$$P = \frac{W - (X_R - X_L) - \frac{c'\ell\sin\alpha}{F} + \frac{u\ell\tan\phi'\sin\alpha}{F}}{m_\alpha} \qquad [6]$$

where:

$$m_\alpha = \cos\alpha + \frac{\sin\alpha\tan\phi'}{F}$$

The factor of safety, F, in [6] can be either with respect to moment or force equilibrium depending upon the factor of safety equation being solved. Any one of a number of possible assumptions can be made in order to compute the interslice shear forces. The various methods of slices that are commonly used can be categorized in terms of the assumption that is made regarding the interslice shear forces, and whether the computed factors of safety satisfy moment or force equilibrium, or both.

Non-Circular Slip Surface

Figure 2 shows the forces that are used in the derivation of the moment and force equilibrium factor of safety equations in the GLE method for a non-circular slip surface. The slip surface that is shown starts and ends with a circular portion, and has a central linear portion. The non-circular portion is assumed to be the result of a geological discontinuity which does not allow the slip surface to penetrate deeper. This type of non-circular surface is termed a composite slip surface (Fredlund, 1975), and it attempts to model typical observed modes of failure (Krahn et al, 1979). From a theoretical standpoint there are two changes in the moment equilibrium equation. The moment arm associated with the mobilized shear force becomes a variable distance, R, and the normal force, P, has an offset arm, f.

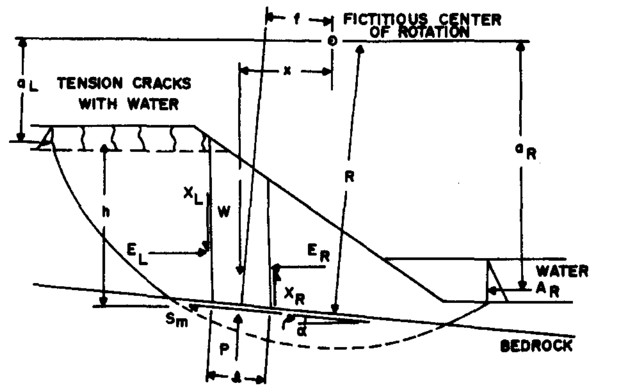

Fig. 2 Forces Acting For The Method Of Slices (Composite Slip Surface)

The factor of safety equation for moment equilibrium becomes,

$$F_m = \frac{\Sigma[c'\ell + (P - u\ell)\tan\phi']R}{\Sigma Wx - \Sigma Pf \pm Aa} \quad [7]$$

The factor of safety equation for force equilibrium [5] and the equation for the normal force [6] remain unchanged. The center for moment equilibrium is the center of rotation for the circular portion of the slip surface.

COMPARISON OF THE GLE FORMULATION AND COMMONLY USED METHODS OF SLICES

Morgenstern-Price Method

Morgenstern and Price (1965) solved for the factor of safety using the summation of forces tangential and normal to the base of a slice and the summation of moments about the center of the base of each slice. The equations were written for a slice of infinitesimal thickness. The force and moment equilibrium equations were combined and a modified Newton-Raphson numerical technique was used to solve for the factor of safety satisfying force and moment equilibrium. The solution required an arbitrary assumption regarding the direction of the resultant of the interslice shear and normal forces. Figure 3 shows typical functional forms which can be written as

$$X/E = \lambda f(x)$$

where:

- $f(x)$ = a function that describes the manner in which X/E varies across the slope, and
- λ = a constant representing the percentage (i.e., portion of the function used when solving for the factor of safety.

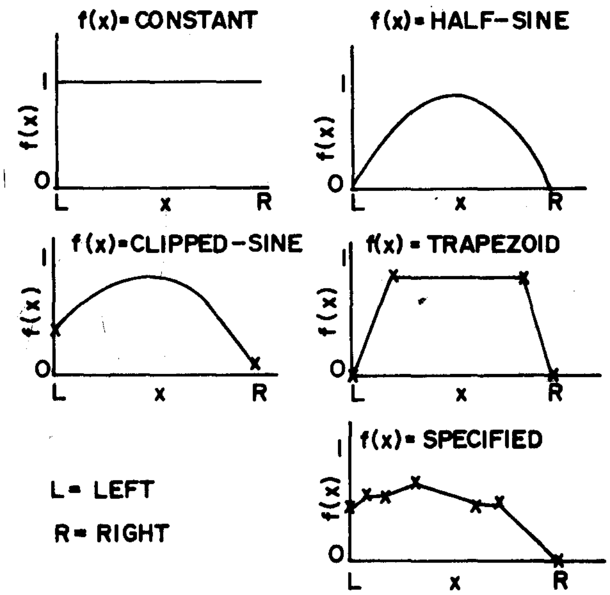

Fig. 3 Typical Functional Variations For The Direction Of The Interslice Force With Respect To The X Direction

The moment and force factor of safety equations (i.e., equations [3] and [5] for the GLE method can be solved independently for a given function, f(x), by assuming various trial values of "λ" (Fredlund, 1974). A best-fit regression line through the factors of safety associated with the "λ" values indicates the factor of safety satisfying both moment and force equilibrium. The factor of safety satisfying both force and moment equilibrium can also be obtained by using a Newton-Raphson numerical solver.

The assumption regarding the interslice forces and the elements of statics used in the Morgenstern-Price formulation are the same as those used in the GLE formulation. However, there is a light difference in the way the normal force is applied to the base of the slice (Figure 4). The Morgenstern-Price method uses integration across the slope, and this results in a linear variation of the normal force across the base of the slice. As a result, the resultant normal force, P, can have a slight offset from the center of the slice. The GLE formulation assumes that the resultant normal force acts through the center of the slice.

Fredlund and Krahn (1977) used a 12.2 m high, 2:1 slope to study the difference in factor of safety and "λ" between the Morgenstern-Price formulation and the GLE formulation. The first example considered a circular slip surface passing through a homogeneous soil with an effective angle of internal friction of 20 degrees and an effective cohesion intercept of 28.7 kPa. The example problem was then modified to form a composite slip surface by introducing a hard layer at a depth of 1.52 m below the toe of the slope. A thin, soft layer with an effective angle of internal friction of 10 degrees and zero effective cohesion was located immediately above the hard layer. Two pore pressure coefficients were used for each case. The computed factors of safety and "λ" values are presented in Table 1. The Morgenstern-Price method was solved using the University of Alberta computer program (Krahn et al, 1971) and the GLE formulation was solved using the SLOPE computer program (Fredlund, 1974).

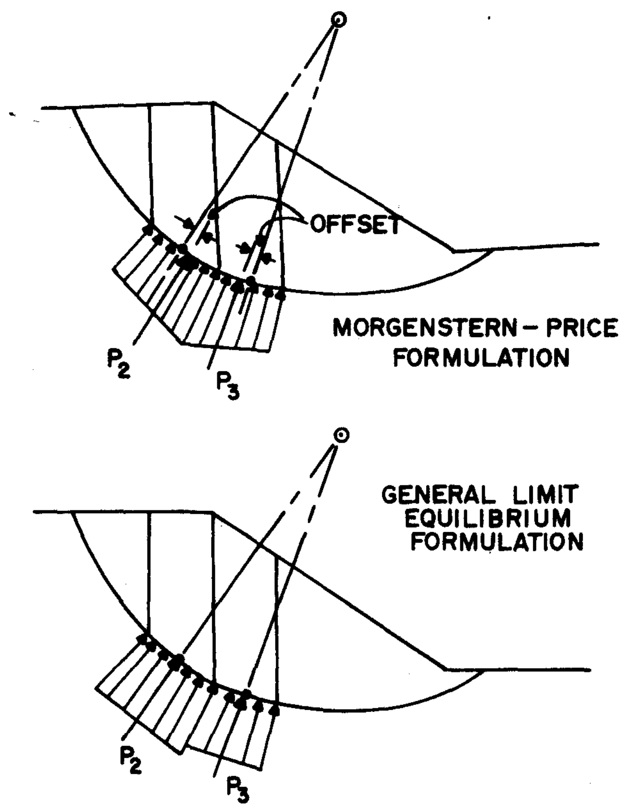

Fig. 4 Point of Application of Normal Force For Morgenstern-Price Formulation and General Formulation

Table I. Comparison of Factors of Safety and "λ" Using the Morgenstern-Price Method and the GLE Method

Shape of Failure Surface	r_u	Side Force Function	Morgenstern-Price Method		GLE Method	
			F*	λ	F*	λ
Circular	0.0	Constant	2.085	0.257	2.076	0.254
Circular	0.0	Half Sine	2.085	0.314	2.076	0.318
Circular	0.25	Constant	1.772	0.351	1.765	0.244
Circular	0.25	Half Sine	1.770	0.434	1.764	0.304
Composite	0.0	Constant	1.394	0.182	1.378	0.159
Composite	0.0	Half Sine	1.386	0.218	1.370	0.187
Composite	0.25	Constant	1.137	0.334	1.124	0.116
Composite	0.25	Half Sine	1.117	0.441	1.118	0.130

* The tolerance is 0.001

The average difference in the factors of safety obtained from the Morgenstern-Price method and the GLE method was negligible (i.e., less than 0.01). The average difference in the "λ" values was in the order of 0.1 with the Morgenstern-Price formulation being higher. The difference in "λ" value is attributed to the procedure used to handle the normal force at the base of the slice. The results indicate that there is a small difference in the computations due to the manner in which this force is handled.

The Spencer Method

The Spencer method assumes a constant relationship between the magnitude of the interslice shear and normal forces (Spencer, 1967).

$$X/E = \tan\theta \qquad [9]$$

where:

θ = angle of the resultant interslice force from the horizontal.

Equation [9] is the same as equation [8] if the interslice force function, f(x), is equal to 1; then "λ" is equal to tan θ. Spencer (1967) summed forces perpendicular to the interslice forces to derive the normal force equation. However, the same equation can be derived by summing forces in a vertical and horizontal direction (i.e., equation [6].

Spencer (1967) derived two factor of safety equations; one satisfying force equilibrium. These equations are essentially the same as those proposed in the GLE formulation when the interslice force function, f(x), is assumed to be a constant.

Two example problems were selected to demonstrate the relationship between the Spencer method and the GLE formulation (Figure 5).

Example No. 1 considers a circular slip surface while example No. 2 is forced into a composite mode by a bedrock layer. The examples are similar to those described previously in this paper (Fredlund and Krahn, 1977) with the exception that a 3.05 m tension crack zone with no water is assumed and the pore pressure coefficient is 0.2.

The results for example No. 1 (Figure 6) show that the Spencer method and the GLE formulation give the same results. Likewise, the results for example No. 2 (Figure 7) show agreement between the Spencer method and the GLE formulation. The interslice force directions on Figures 6 and 7 are presented in terms of tan θ (i.e., λ).

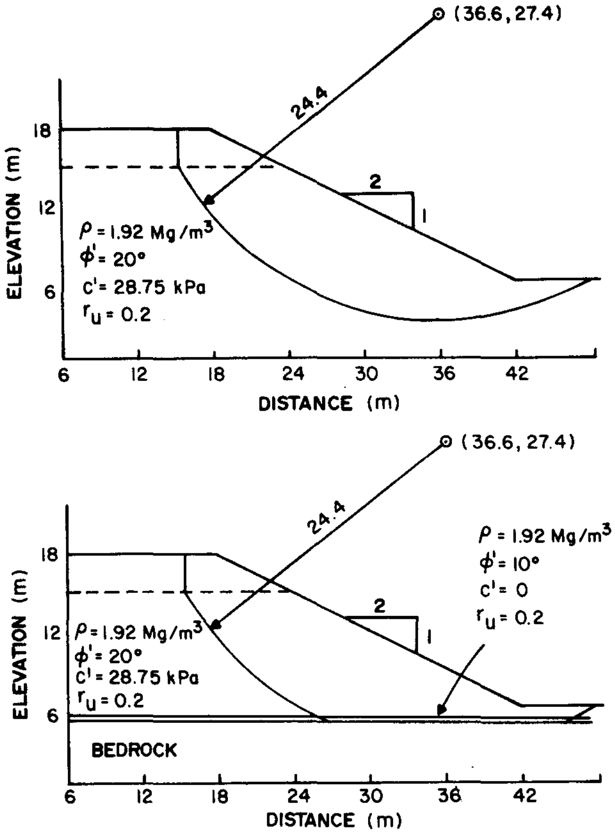

Fig. 5 Example Problems With Circular And Composite Slip Surfaces

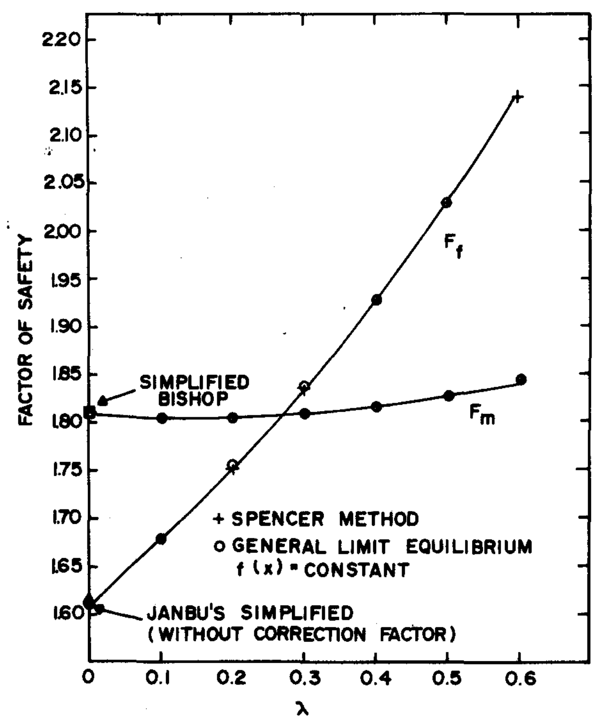

Fig. 6 Comparison of Factors Of Safety For Example No. 1 (Circular Slip Surface)

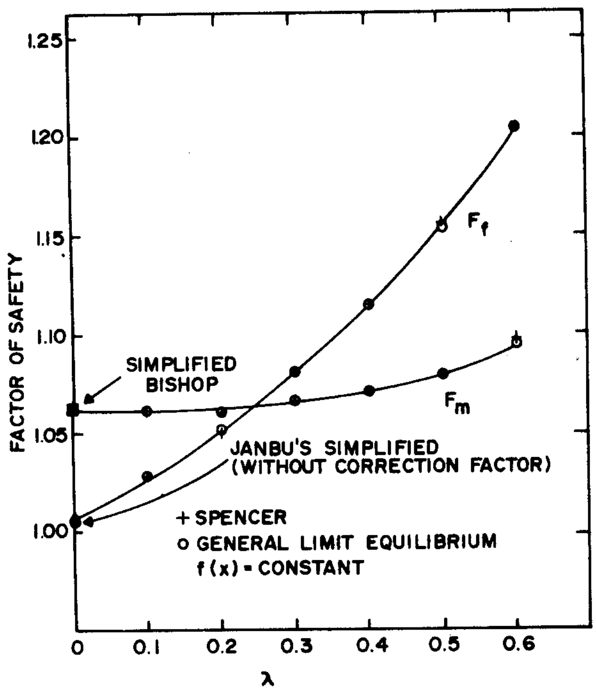

Fig. 7 Comparison Of Factors Of Safety For Example No. 2 (Composite Slip Surface)

Simplified Bishop Method

The simplified Bishop method neglects the interslice shear forces (Bishop, 1955). The normal force equation is the same as equation [6] with the interslice shear forces set to zero. The factor of safety equation is derived by taking moments about the center of rotation. In other words, the simplified Bishop method corresponds to the moment equilibrium factor of safety equation [3] when "λ" is equal to zero or the Spencer moment equilibrium equation when θ is equal to zero (See Figure 6 and 7). The simplified Bishop method bears the same relationship to the GLE formulation (or the Spencer equations) regardless of whether the slip surface is circular or composite. In general, the difference between the simplified Bishop factor of safety and the factor of safety satisfying both force and moment equilibrium, decreases as a particular slip surface has an increasing planar portion.

Janbu Simplified Method

In the derivation of the Janbu simplified method the interslice shear forces are assumed to be zero (Janbu et al, 1956). The normal force equation is the same as equation [6] with the interslice shear forces set to zero. The factor of safety is computed from the horizontal force equilibrium equation (i.e., equation [5]). Then an empirical correction factor is multiplied by the computed factor of safety in an attempt to account for the effect of the interslice shear forces. The empirical correction factor is related to the shear strength properties and the shape of the slip surface. Moment equilibrium is not satisfied. The uncorrected factors of safety correspond to the force equilibrium factor of safety, [5], when "λ" equals zero. For examples No. 1 and No. 2, the uncorrected factors of safety for the Janbu simplified method are 1.609 and 1.005, respectively, (See Figures 6 and 7). The empirical correction factor generally increases the factor of safety by up to approximately 10 percent.

Janbu Generalized Method

The Janbu generalized method includes the effect of interslice forces by making an assumption regarding the point at which the interslice forces act (i.e., the line of thrust; Janbu, 1954; Janbu et al, 1956). The normal force equation is derived from the summation of vertical forces equation [6].

The factor of safety equation is derived from the horizontal force equilibrium equation (equation [5]). In order to solve for the factor of safety, the interslice shear forces are computed from the summation of the moments about the center of the base of each slice.

$$X_R = E_R \tan \alpha_t - (E_R - E_L)\, t_R/b \qquad [10]$$

where:

α_t = angle between the line of thrust on the right side of a slice and the horizontal.

t_R = vertical distance from the base of the slice to the line of thrust on the right side of the slice.

The horizontal interslice forces required for equation [10], are obtained by summing forces in the horizontal direction on each slice.

$$E_L - E_R = S_m \cos \alpha - P \sin \alpha \qquad [11]$$

Once the Janbu generalized factor of safety equation, [5], has been solved, it is possible to plot the computed interslice shear and normal forces and determine a corresponding side force function. It was not possible to obtain a side force function for examples No. 1 and No. 2 since convergence difficulty was encountered because of the geometry that was arbitrarily chosen in these two cases.

Example No. 3 (Figure 8) is used to demonstrate the relationship between the Janbu generalized method and the GLE formulation. The factor of safety by the Janbu generalized method is 1.19.

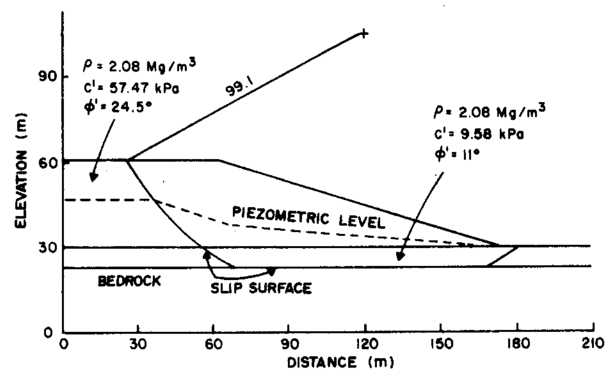

Fig. 8 Example No. 3 To Demonstrate The Janbu Generalized Method

The ratio of the computed interslice shear and normal force was plotted versus the distance along the slip surface (Figure 9). The resulting plot was used as a side force function, f(x), in the GLE equations with the "λ" value set to 1. The factor of safety from the GLE (Force) equation, [5], yielded the same factor of safety as that obtained by the Janbu generalized method. In other words, the summation of forces on each slice can be viewed as a means of obtaining a particular type of side force function. In this way, moment equilibrium is implicitly satisfied.

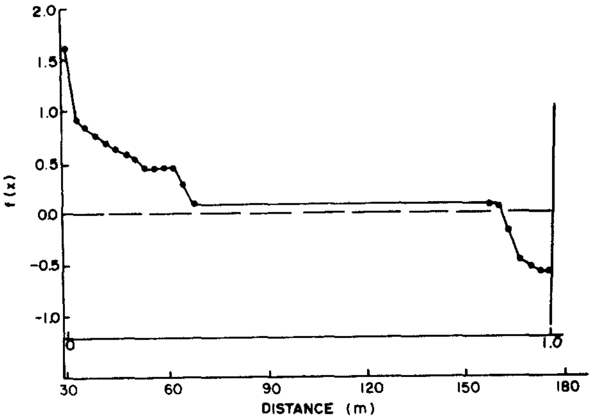

Fig. 9 Side Force Function For Example No. 3 Using The Janbu Generalized Method

Force Equilibrium Methods

- Lowe and Karafiath Method

The Lowe and Karafiath method computes the factor of safety from a force equilibrium equation. The direction of the resultant of each interslice force is assumed to be equal to the average of the surface and slip surface slopes. The computed side force functions for examples No. 1 and 2 (Figure 5) are shown in **Figures 10 and 11**, respectively. These side force functions were then used in the GLE formulation with "λ" set to 1.

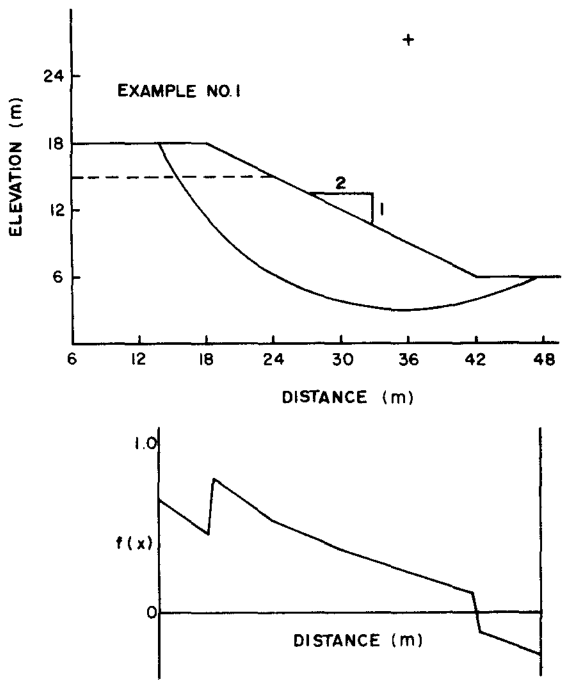

Fig. 10 Side Force Function Using Lowe and Karafiath Method (Example No. 1)

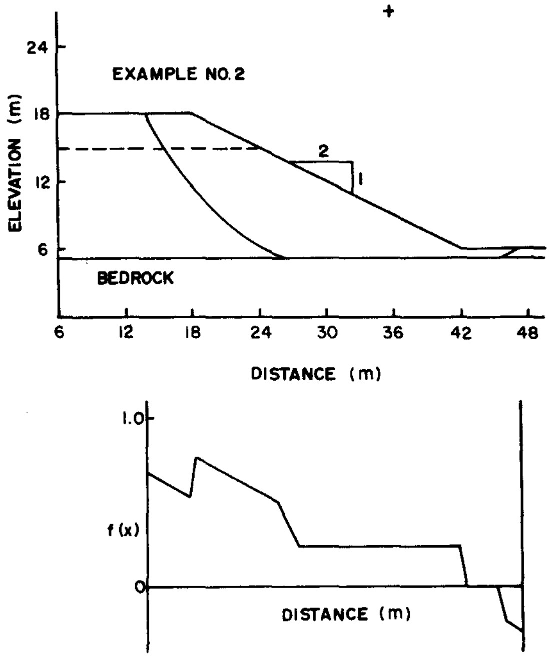

Fig. 11 Side Force Function Using Lowe And Karafiath Method (Example No. 2)

Table 2 shows the force and moment equilibrium factors of safety from the GLE equations when using the Lowe and Karafiath assumption regarding the interslice force directions.

Table II. Factors of Safety Using the Lowe and Karafiath Interslice Force Assumption

Example No.	GLE Method	
	F_f	F_m
1	1.880	1.791
2	1.096	1.068

The force equilibrium factor of safety, [5], corresponds to the Lowe and Karafiath factor of safety. These factors of safety can be compared with those obtained by other methods (See Figure 6 and 7). The Lowe and Karafiath assumption regarding the interslice forces can be interpreted as a special side force function applied to the GLE formulation.

- Corps of Engineers Method

The Corps of Engineers method (1970) computes the factor of safety from the force equilibrium equation [5]. The direction of the resultant interslice force is assumed to be equal to the average surface slope. This appears to be interpreted as either equal to the average slope between the extreme extrance and exit of the failure surface, (Case 1) or the changing slope of the ground surface (Case 2). A side force function is computed for Case 1 and 2, (Figure 12).

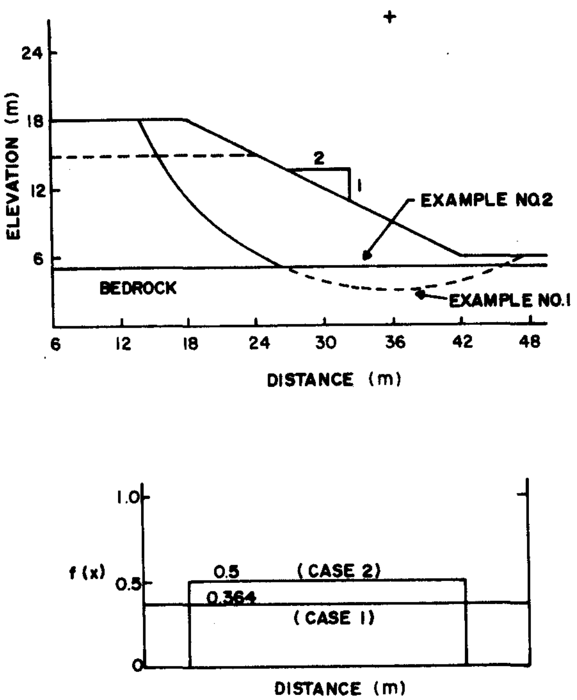

Fig. 12 Side Force Function Using the Corps Of Engineers Method (Example No. 1 and 2)

The side force functions, f(x), are the same in both examples No. 1 and 2. Table 3 summarizes the factors of safety obtained using the GLE equations along with the Corps of Engineers interslice force assumptions. The "λ" value is set equal to 1.

Table III. Factors of Safety Using the Corps of Engineers Interslice Force Assumptions

Example No.	Case No.	GLE Method F_f	GLE Method F_m
1	1	1.893	1.810
	2	2.000	1.801
2	1	1.102	1.068
	2	1.100	1.050

The force equilibrium factors of safety (GLE method) correspond to the Corps of Engineers factor of safety. Since the function, f(x), is a constant, it can be set to 1 and "λ" can be used as before. The factors of safety for the Case 1 assumption can be seen to correspond to a "λ" value of 0.364 in Figure 6 and 7. Again, the Corps of Engineers assumption regarding the direction of the resultant interslice forces can be interpreted as a special side force function applied to the GLE formulation.

The Corps of Engineers factors of safety (as well as the Lowe and Karafiath factors of safety) lie along a force equilibrium factor of safety line. The magnitude of the factor of safety may either be higher or lower than the factor of safety satisfying both force and moment equilibrium. It should be noted that the force equilibrium factor of safety is more highly influenced by the side force assumption than the moment equilibrium factor of safety.

SUMMARY

The factor of safety equations for all methods of slices considered can be written in the same form if the moment and/or force equilibrium are explicity satisfied. This applies for both circular and composite (non-circular) slip surfaces. The normal force at the base of each slice can be solved using the same equation for all methods, with the exception of the Ordinary method. The type of side force function assumed or computed, results in a variation in the normal force at the base of a slice. This also accounts for the difference in factors of safety between various methods when force or moment equilibrium are considered independently. The analytical aspects of slope stability analysis can be viewed in terms of one factor of safety equation satisfying overall moment equilibrium and another satisfying overall force equilibrium for various "X/E" values. Each of the methods of slices considered becomes a special case of the proposed GLE formulation.

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