One-dimensional consolidation theory: unsaturated soils

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A one-dimensional consolidation theory is presented for unsaturated soils. The assumptions made are in keeping with those used in the conventional theory of consolidation for saturated soils, with the additional assumption that the air phase is continuous. Two partial differential equations are derived to describe the transient processes taking place as a result of the application of a total load to an unsaturated soil.

After a load has been applied to the soil, air and water flow simultaneously from the soil until equilibrium conditions are achieved. The simultaneous solution of the two partial differential equations gives the pore-air and pore-water pressures at any time and any depth throughout the soil. Two families of dimensionless curves are generated to show the pore-air and pore-water dissipation curves for various soil properties.

For the case of an applied total load, two equations are also derived to predict the initial pore-air and pore-water pressure boundary conditions. An example problem demonstrates the nature of the results.

L'article présente une théorie de consolidation unidimensionnelle pour les sols non saturés. Les hypothèses faites sont conformes à celles utilisées dans la théorie conventionnelle pour les sols saturés, avec l'hypothèse additionnelle de la continuité de la phase gazeuse. Deux équations aux dérivées partielles sont établies pour décrire les phénomènes transitoires qui résultent de l'application d'une charge à un sol non saturé.

Après l'application d'une charge au sol, l'eau et l'air s'échappent simultanément du sol jusqu'à ce que des conditions d'équilibre soient atteintes. La solution simultanée des deux équations aux dérivées partielles donne les pressions interstitielles de l'air et de l'eau en tout temps et à toute profondeur dans le sol. Deux familles de courbes adimensionnelles sont générées pour donner les courbes de dissipation des pressions d'air et d'eau pour différentes propriétés du sol.

Pour le cas d'une charge totale appliquée, deux équations sont également établies pour prédire les pressions initiales dans l'air et dans l'eau. Un cas type démontre la nature des résultats.

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Introduction

Since the inception of modern soil mechanics, Terzaghi's theory of consolidation for saturated soils has formed an extremely useful conceptual framework in geotechnical engineering. Unfortunately, the study of the behavior of unsaturated soils has taken place in the absence of a similar theoretical framework. As a result, it has been difficult to envisage the transitions in theory when going from a saturated soil to an unsaturated soil.

As far back as 1941, Biot presented an analysis of the transient flow problem in saturated soils. He suggested two constitutive relations for the soil and solved for changes in the pore-water pressure with time. Biot considered the air to be in an occluded state with no flow of air during the consolidation process.

In 1965, Barden presented an analysis of the one-dimensional consolidation of compacted, unsaturated clay. He subdivides the consolidation problem into various categories depending upon the degree of saturation of the soil. He states the problem is indeterminate and, therefore, various assumptions are made in order to complete the analysis. Bishop's equation is used to describe the stress conditions in the unsaturated soil.

Partial differential equations have been developed in the soil science discipline to describe unsteady moisture movement. More recently these equations have received increasing acceptance in the soil mechanics field (Aitchison *et al.* 1965). These equations should be considered as a special case since the compressibility of the soil structure and the escape of air are not taken into consideration. They are generally not applied to transient processes associated with the application of an external load.

Fredlund and Morgenstern (1977) proposed stress state variables for unsaturated soils on the basis of the equilibrium equations for a multiphase system. These were also verified experimentally. The element of unsaturated soil was considered as a four phase system with two phases that come to equilibrium

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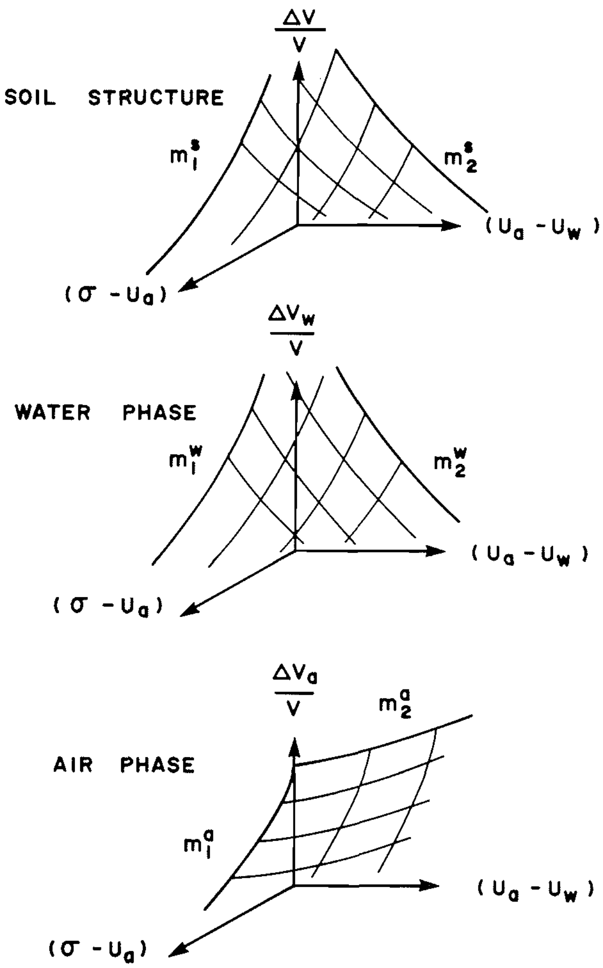

FIG. 1. Constitutive surfaces for the various phases of an unsaturated soil.

under applied stress gradients (i.e., soil particles and contractile skin or air–water interface) and two phases that flow under applied stress gradients (i.e., air and water). Fredlund and Morgenstern (1976) also proposed and experimentally tested constitutive relations for volume change in unsaturated soils. In addition, the continuity requirements for an element of unsaturated soil were outlined. The description of the stress, continuity, and constitutive relations, and suitable flow and compressibility laws for air and water, provide the necessary physical requirements for a more rigorous formulation of transient processes in unsaturated soils.

This paper presents a general one-dimensional consolidation (or swelling) theory for unsaturated soils within a theoretical framework similar to that for saturated soils. As well, equations are derived to predict the initial air and water boundary conditions associated with the application of an externally applied load. An example problem is included to demonstrate the solution of the above equations.

Physical Requirements for the Formulation

The state of stress in an unsaturated soil can be described by any two of a possible three stress state variables (Fredlund and Morgenstern 1977). Acceptable combinations are: (1) $(\sigma - u_a)$ and $(u_a - u_w)$; (2) $(\sigma - u_w)$ and $(u_a - u_w)$; and (3) $(\sigma - u_a)$ and $(\sigma - u_w)$. The stress variables selected to derive the consolidation equations in this paper are $(\sigma - u_a)$ and $(u_a - u_w)$, where σ = total stress; u_a = pore-air pressure; and u_w = pore-water pressure. Continuity of an unsaturated soil element requires that the overall volume change of the element must equal the sum of the volume changes associated with the component phases (Fredlund 1973). If the soil particles are considered incompressible and the volume change of the contractile skin (i.e., air–water interface) is considered as internal to the element, the continuity requirement can be written:

[1] $\Delta V/V = \Delta V_w/V + \Delta V_a/V$

where V = overall volume of the soil element; V_w = volume of water in the soil element; and V_a = volume of air in the soil element.

If any two of the volume changes are known, the third can be computed. In other words, it is necessary to have two constitutive equations to define volume change behavior in unsaturated soils.

Fredlund and Morgenstern (1976) proposed and tested constitutive relations to link the stress and deformation state variables. The proposed constitutive relationship for the soil structure is given by [2] and the relationship for the water phase is given by [3] (Fig. 1).

[2] $\Delta V/V = m_1^s d(\sigma - u_a) + m_2^s d(u_a - u_w)$

[3] $\Delta V_w/V = m_1^w d(\sigma - u_a) + m_2^w d(u_a - u_w)$

where m_1^s = compressibility of the soil structure when $d(u_a - u_w)$ is zero; m_2^s = compressibility of the soil structure when $d(\sigma - u_a)$ is zero; m_1^w = slope of the $(\sigma - u_a)$ plot when $d(u_a - u_w)$ is zero; and m_2^w = slope of the $(u_a - u_w)$ plot when $d(\sigma - u_a)$ is zero.

The constitutive relationship for the air phase is the difference between [2] and [3] because of the continuity requirement.

[4] $\Delta V_a/V = m_1^a d(\sigma - u_a) + m_2^a d(u_a - u_w)$

where m_1^a = slope of the $(\sigma - u_a)$ plot when $d(u_a - u_w)$ is zero, and m_2^a = slope of the $(u_a - u_w)$ plot when $d(\sigma - u_a)$ is zero.

Flow of the water phase is described by Darcy's law (Childs and Collis-George 1950).

[5] $v = (-k_w/\gamma_w)(\partial u_w/\partial y)$

where v = water velocity; k_w = coefficient of permeability with respect to the water phase; γ_w = density of water; and y = depth in the y-direction.

Flow of the air phase is described by Fick's law (Blight 1971).

[6] $v_a = -D(\partial p/\partial y)$

where v_a = mass rate of air flow; D = a transmission constant having the same units as coefficient of permeability; p = absolute air pressure (i.e., $u_a + u_{atm}$); and u_{atm} = atmospheric air pressure.

The isothermal compressibility equation for the air phase, β_a, is (Fredlund 1976):

[7] $\beta_a = 1/(u_a + u_{atm})$

and the isothermal compressibility equation of an air–water mixture (with no diffusion) in the presence of a particulate mass, β_m, is:

[8] $\beta_m = S\beta_w + B_{aw}(1 - S)/(u_a + u_{atm})$

where S = initial degree of saturation, and B_{aw} = pore-pressure coefficient equal to $\Delta u_a/\Delta u_w$. The above listed physical relationships are sufficient to derive the one-dimensional consolidation (or swelling) equations and the pressure boundary condition equations for an unsaturated soil.

Derivation of the Consolidation Equations

The one-dimensional consolidation equations for unsaturated soils are derived using the conventional assumptions for Terzaghi's consolidation theory with the following additions:

(i) The air phase is continuous.

(ii) The coefficients of permeability with respect to water and air, and the volume change moduli remain constant during the transient processes.

(iii) The effects of air diffusing through water and the movement of water vapor are ignored.

The above assumptions are not completely accurate for all cases; however, they are reasonable for a first attempt to derive a general consolidation theory for unsaturated soils.

After applying a load to an unsaturated soil, there will be a dissipation of the excess pore-air and pore-water pressures. In order to compute these values as a function of time, it is necessary to have two equations. This is accomplished by independently considering the continuity of the water and air phases. Then the derived equations are solved simultaneously to give the water and air pressures at any elapsed time.

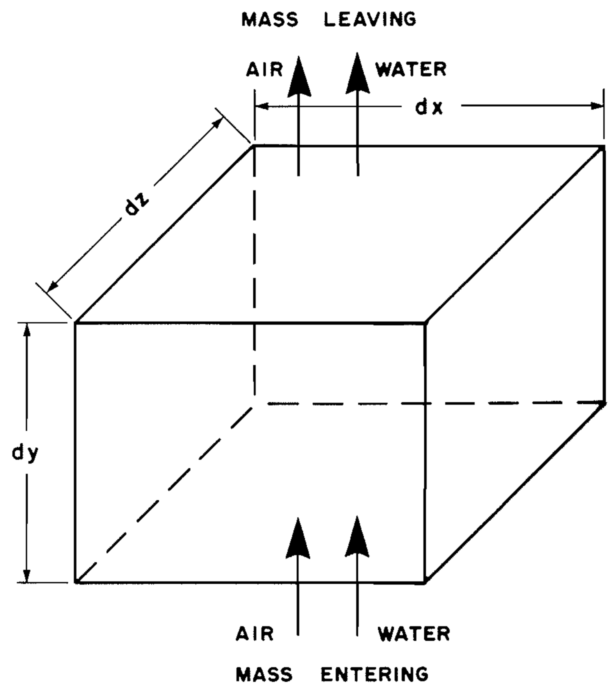

FIG. 2. A referential element in the soil mass.

Water Phase Partial Differential Equation

Let us consider a referential soil element as shown in Fig. 2. The water phase is assumed incompressible. For the consolidation process, water flows out of the element with time. The constitutive relationship for the water phase defines the volume of water in the element for any combination of total, air, and water pressures. The volume of water entering and leaving the element in the y-direction is described by Darcy's law as:

[9] Volume entering $= (-k_w/\gamma_w)(\partial u_w/\partial y)\,dx\,dz$

The net flux of water in the element is:

[10] $\partial(\Delta V_w/V)/\partial t = (-k_w/\gamma_w)(\partial^2 u_w/\partial y^2)$

Equation [10] can be equated to the constitutive relationship for the water phase ([3]) in accordance with the continuity requirement ([1]).

[11] $$m_1^w \frac{\partial(\sigma - u_a)}{\partial t} + m_2^w \frac{\partial(u_a - u_w)}{\partial t} = \frac{-k_w}{\gamma_w}\frac{\partial^2 u_w}{\partial y^2}$$

For the consolidation process the change in total stress with respect to time is set to zero. Simplifying and rearranging [11], the water phase partial differential equation can be written:

[12] $\partial u_w/\partial t = -C_w(\partial u_a/\partial t) + c_v^w(\partial^2 u_w/\partial y^2)$

where $C_w = (1 - m_2^w/m_1^w)/(m_2^w/m_1^w)$ and is called the interactive constant associated with the water phase equation. This equation may be further simplified by defining R_w as m_2^w/m_1^w. When the soil is saturated, $R_w = 1$. $c_v^w = (1/R_w)(k_w/\gamma_w m_1^w)$ and is called the coefficient of consolidation with respect to the water phase.

As a soil becomes saturated, the interactive constant approaches zero and [12] reverts to Terzaghi's one-dimensional consolidation equation. The fact that the air phase cannot remain continuous as a soil goes towards saturation is a physical restriction; however, it does not pose a mathematical problem. Equation [12] also reverts to Terzaghi's equation whenever the air pressure induced is small.

Air Phase Partial Differential Equation

The air phase is compressible and flows independent of the water phase when subjected to an air pressure gradient. As well, the constitutive relationship for the air phase defines the volume of air in the element for any combination of the total, air, and water pressures. According to Fick's law the mass of air entering the element in the y-direction is:

$$[13] \quad \text{Mass entering} = -D(\partial p/\partial y)\,dx\,dz$$

The net mass flux of air in the element is:

$$[14] \quad \partial m/\partial t = -D(\partial^2 p/\partial y^2)$$

where m = mass of air in the element.

The mass rate of change is written in terms of a volume rate of change by differentiating the relationship between mass and volume.

$$[15] \quad \partial(V_a/V)/\partial t = \partial(m/\gamma_a)/\partial t$$

For isothermal conditions the density of air, γ_a, is:

$$[16] \quad \gamma_a = (w/R\theta)p$$

where w = molecular weight of the mass of air; R = universal gas constant; and θ = absolute temperature.

The mass of air is written in terms of the density of air, the degree of saturation, S, and the porosity of the soil, n.

$$[17] \quad m = (1 - S)n\gamma_a$$

Substituting [14], [16], and [17] into [15] gives:

$$[18] \quad \frac{\partial(V_a/V)}{\partial t} = \frac{-DR\theta}{wp}\frac{\partial^2 u_a}{\partial y^2} + \frac{(1-S)n}{p}\frac{\partial u_a}{\partial t}$$

Equating [18] to the constitutive relationship for the air phase (i.e., [4]), gives:

$$[19] \quad m_1^a \frac{\partial(\sigma - u_a)}{\partial t} + m_2^a \frac{\partial(u_a - u_w)}{\partial t} = \frac{-DR\theta}{wp}\frac{\partial^2 u_a}{\partial y^2} + \frac{(1-S)n}{p}\frac{\partial u_a}{\partial t}$$

The change in total stress with respect to time can be set to zero for the consolidation process. Simplifying and rearranging [19], the air phase partial differential equation can be written as follows.

$$[20] \quad \partial u_a/\partial t = -C_a(\partial u_w/\partial t) + c_v^a(\partial^2 u_a/\partial y^2)$$

where

$$C_a = \frac{m_2^a/m_1^a}{(1 - m_2^a/m_1^a) + \dfrac{(1-S)n}{(u_a + u_{atm})m_1^a}}$$

and is called the interactive constant associated with the air phase equation. This equation may be further simplified by defining R_a as m_2^a/m_1^a.

$$c_v^a = \frac{DR\theta}{w}\frac{1}{(1 - R_a)(u_a + u_{atm})m_1^a + (1-S)n}$$

and is called the coefficient of consolidation with respect to the air phase.

As a soil becomes completely dry, the interactive constant approaches zero and [20] reverts to the form presented by Blight (1971).

The dissipation of the excess pressure of the pore-air and pore-water phases is obtained by a simultaneous solution of [12] and [20] using the finite difference technique described in the Appendix.

The results can be expressed in a dimensionless form by defining an average degree of consolidation and time factor for each of the fluid phases. The average degree of consolidation for the water phase is:

$$[21] \quad U_w = 1 - \frac{\int_0^{2H} u_w\,dy}{\int_0^{2H} u_{wi}\,dy}$$

where U_w = average degree of consolidation with respect to the water phase; u_{wi} = initial water pressure; u_w = water pressure at any time; and H = length of drainage path.

The time factor for the water phase is:

$$[22] \quad T_w = c_v^w t/H^2$$

where t = elapsed time.

Similarly, the average degree of consolidation and

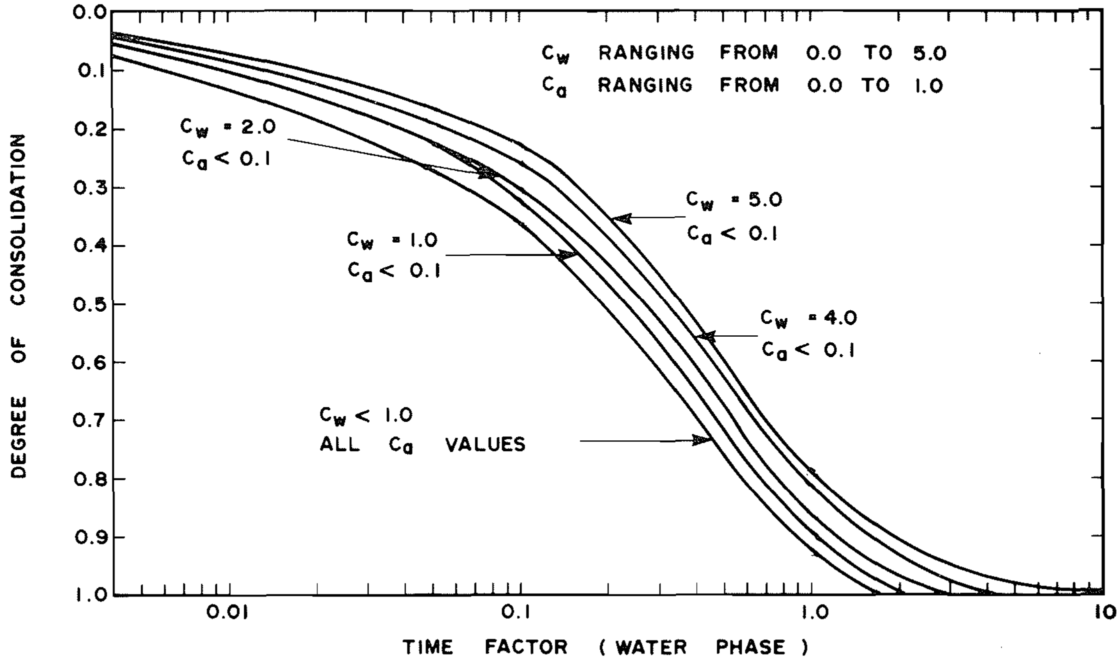

FIG. 3. Dimensionless time factor versus degree of consolidation curves for the water phase.

the dimensionless time factor with respect to the air phase are defined as:

$$[23]\quad U_a = 1 - \frac{\int_0^{2H} u_a \, dy}{\int_0^{2H} u_{ai} \, dy}$$

and

$$[24]\quad T_a = c_v{}^a t / H^2$$

where U_a = average degree of consolidation with respect to the air phase; T_a = time factor with respect to the air phase; u_{ai} = initial air pressure; and u_a = air pressure at any time.

Figure 3 shows the water phase degree of consolidation versus time factor curves for various air–water interaction constants. Similar curves for the air phase are shown in Fig. 4. The interactive constant in the air phase partial differential equation was assumed to be constant for the calculation of pore-air pressure dissipation. The curves cover anticipated reasonable ranges for the soil moduli. They also show a smooth transition towards the case of a completely saturated soil (Terzaghi 1936) and the case of a completely dry soil (Blight 1971).

Varying Permeabilities During Consolidation

The partial differential equation for the pore-water and pore-air phases can also be derived for the case where the coefficients of permeability are variables during the process. The derivations follow a form similar to that presented above but the coefficients of permeability are treated as variables during differentiation. The pore-water partial differential equation now becomes:

$$[25]\quad \frac{\partial u_w}{\partial t} = -C_w \frac{\partial u_a}{\partial t} + c_{v_1}{}^w \left[k_w \frac{\partial^2 u_w}{\partial y^2} + \frac{\partial u_w}{\partial y} \frac{\partial k_w}{\partial y} \right]$$

where $c_{v_1}{}^w = 1/R_w \gamma_w m_1{}^w$.

The pore-air partial differential equation becomes

$$[26]\quad \frac{\partial u_a}{\partial t} = -C_a \frac{\partial u_w}{\partial t} + c_{v_1}{}^a \left[(u_a + u_{atm}) \left[k_a \frac{\partial^2 u_a}{\partial y^2} + \frac{\partial u_a}{\partial y} \frac{\partial k_a}{\partial y} \right] + k_a \frac{\partial u_a}{\partial y^2} \right]$$

where

$$c_{v_1}{}^a = 1/[(1 - R_a)(u_a + u_{atm}) m_1{}^a + (1 - S)n]$$

These equations can readily be solved using a finite difference technique; however, it is difficult to present the solutions in dimensionless form due to nonlinearity.

Pore Pressure Boundary Condition Equations

When an external load is applied to an element of

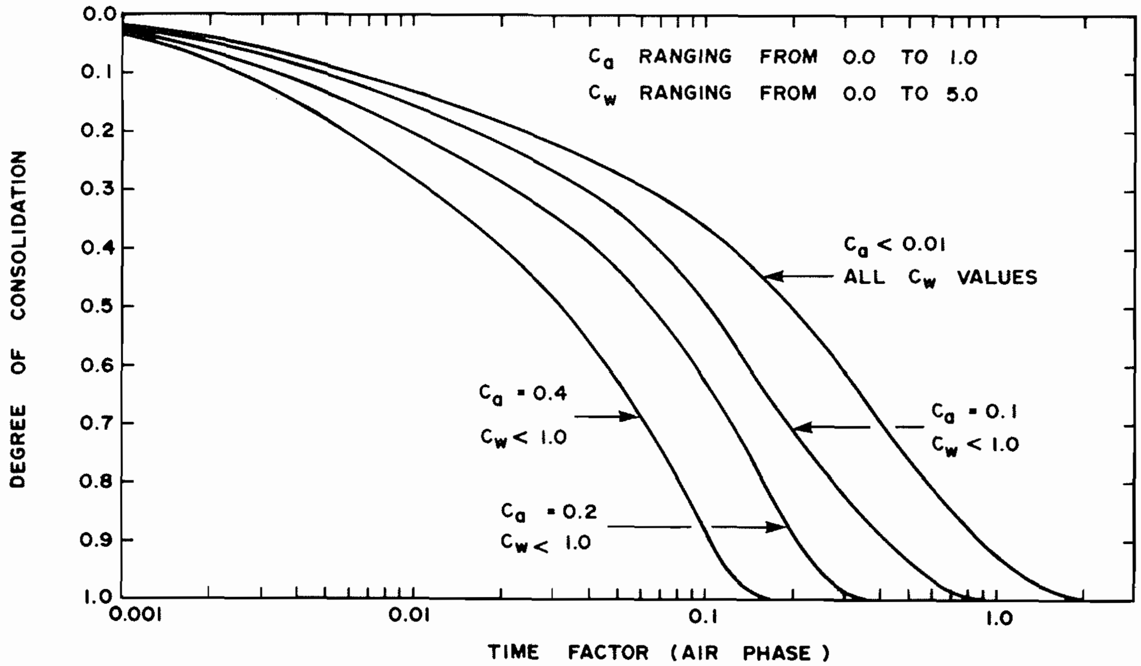

FIG. 4. Dimensionless time factor versus degree of consolidation curves for the air phase.

unsaturated soil, instantaneous compression occurs under undrained conditions and excess pressures are induced in the air and water phases. Two equations are necessary to predict the relative magnitudes of the excess pore-air and pore-water pressures. The pore pressures depend upon the compressibility of the soil structure, the air and water phases. In addition, the contractile skin has an effect on the relative changes in the pore-air and pore-water pressures. The induced pore pressures form the boundary conditions for the consolidation process.

The overall continuity requirement for the soil requires that the compression of the soil structure must equal the compression associated with the pore-fluid phases. This is satisfied by equating [2] and [8] and using the water phase as the reference phase for the pore-fluid pressure (Fredlund 1976).

$$[27]\quad m_1^s\Delta(\sigma - u_a) + m_2^s\Delta(u_a - u_w) = \left[S\beta_w + \frac{\Delta u_a}{\Delta u_w}\frac{(1-S)}{(u_a + u_{atm})}\right]n\Delta u_w$$

Defining $R_s = m_2^s/m_1^s$, [27] can be solved for the change in pore-water pressure.

$$[28]\quad \Delta u_w = \left[\frac{(R_s - 1) - \dfrac{(1-S)n}{(u_a + u_{atm})m_1^s}}{R_s + Sn\beta_w/m_1^s}\right]\Delta u_a + \left[\frac{1}{R_s + Sn\beta_w/m_1^s}\right]\Delta\sigma$$

Equation [28] involves two unknowns (i.e., Δu_w and Δu_a) and therefore another equation is required for its solution. A second equation is also logical since only one of two necessary constitutive relations has been used in formulating [28]. The second constitutive relation can be incorporated by considering the continuity of the air phase. Volume change described by the compression of the air phase must equal the volume change defined by the air phase constitutive relationship.

$$[29]\quad m_1^a\Delta(\sigma - u_a) + m_2^a\Delta(u_a - u_w) = [(1 - S)n/(u_a + u_{atm})]\Delta u_a$$

Simplifying [29] and solving for the change in pore-air pressure gives:

$$[30]\quad \Delta u_a = \left[\frac{R_a}{(R_a - 1) - \dfrac{(1-S)n}{(u_a + u_{atm})m_1^a}}\right]\Delta u_w - \left[\frac{1}{(R_a - 1) - \dfrac{(1-S)n}{(u_a + u_{atm})m_1^a}}\right]\Delta\sigma$$

Equations [28] and [30] can be solved for the changes in pore-air and pore-water pressures resulting from a change in the applied load. Let us simplify these equations by defining the following variables.

$$[31]\quad R_1 = \frac{(R_s - 1) - \dfrac{(1-S)n}{(u_a + u_{atm})m_1^s}}{R_s + Sn\beta_w/m_1^s}$$

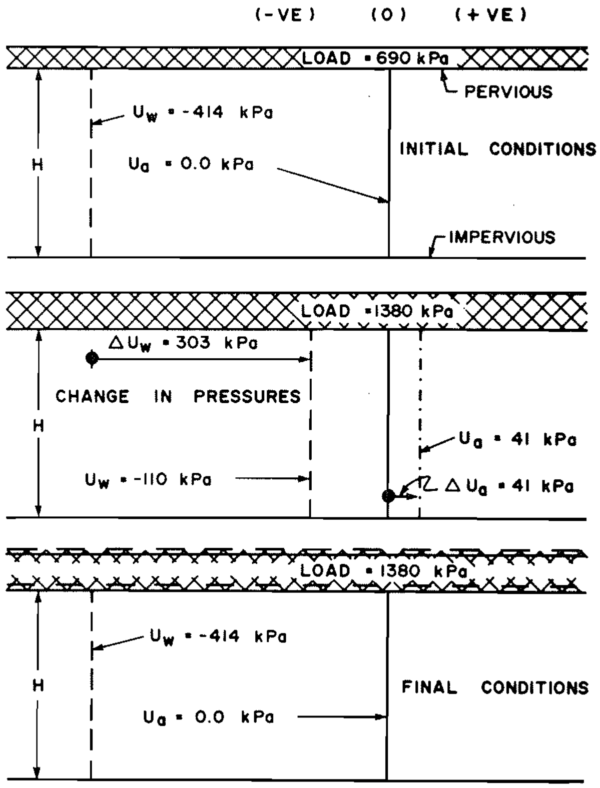

FIG. 5. Initial and final boundary conditions for a consolidation process.

$$R_2 = 1/(R_s + Sn\beta_w/m_1^s) \quad [32]$$

$$R_3 = \frac{R_a}{(R_a - 1) - \dfrac{(1 - S)n}{(u_a + u_{atm})m_1^a}} \quad [33]$$

$$R_4 = \frac{1}{(R_a - 1) - \dfrac{(1 - S)n}{(u_a + u_{atm})m_1^a}} \quad [34]$$

Therefore, [28] and [30] can be written:

$$\Delta u_w = R_1\Delta u_a + R_2\Delta\sigma \quad [35]$$

$$\Delta u_a = R_3\Delta u_w - R_4\Delta\sigma \quad [36]$$

Equations [35] and [36] are combined and solved to give two pore pressure coefficients that can be used to compute the boundary conditions.

$$B_w = \Delta u_w/\Delta\sigma = (R_2 - R_1R_4)/(1 - R_1R_3) \quad [37]$$

$$B_a = \Delta u_a/\Delta\sigma = (R_2R_3 - R_4)/(1 - R_1R_3) \quad [38]$$

Equations [37] and [38] require an iterative technique in their solution since R_1, R_3, and R_4 contain the absolute pore-air pressure value.

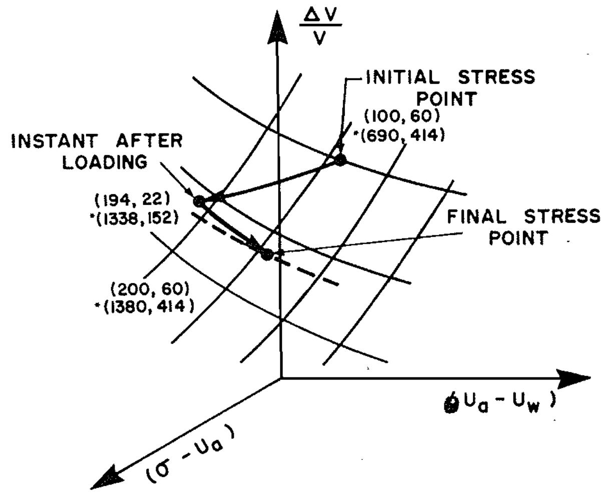

FIG. 6. Initial and final stress points on a constitutive surface for the consolidation process. * kPa.

Example Problem

An example problem is solved to demonstrate the solution of boundary condition equations (i.e., [37] and [38]) and the partial differential equations for the pore-air and pore-water phases (i.e., [12] and [20]). Let us assume that a compacted soil layer overlying an impervious surface is initially in equilibrium with the stresses shown in Fig. 5. The initial equilibrium conditions are altered by changing the total stress applied to the top of the soil. The instantaneous change in total stress of 689 kPa (100 psi) produced a corresponding pore-air and pore-water pressure change of 41.4 kPa (6 psi) and 303 kPa (44 psi), respectively. The initial and final stresses (and stress state variables) with the assumed soil properties are shown in Table 1. The corresponding stress state variable changes on the constitutive surface are shown in Fig. 6. The dissipation of the excess pore-water and pore-air pressures is obtained by simultaneous solution of [12] and [20].

The average degree of consolidation with respect to the water phase (U_w) is plotted against the time factor for water phase (T_w) in Fig. 7. It shows that the U_w versus T_w plot is only slightly different than the conventional Terzaghi consolidation plot for saturated soils. This deviation of the U_w versus T_w plot from the conventional Terzaghi plot can be attributed to the interaction effect in the simultaneous solution of the two partial differential equations.

The average degree of consolidation with respect to the air phase (U_a) plotted against a time factor with respect to the air phase (T_a) is presented in

TABLE 1. Initial and final stress conditions in the soil

Stresses	σ (psi (kPa))	u_a (psi* (kPa))	u_w (psi (kPa))	$(\sigma - u_a)$ (psi (kPa))	$(u_a - u_w)$ (psi (kPa))
Initial condition	100.0 (690)	0.0 (0.0)	−60.0 (−414)	100.0 (690)	60.0 (414)
Instant after loading					
$\Delta u_a = +6$ psi (41 kPa); $\Delta u_w = +44$ psi (303 kPa)	200.0 (1380)	6.0 (41)	−16.0 (−110)	194.0 (1338)	22.0 (152)
Final conditions	200.0 (1380)	0.0 (0.0)	−60.0 (−414)	200.0 (1380)	60.0 (414)

NOTES: S = 50%; n = 50%; m_1^a = 0.0008 in.²/lb (1.16 × 10^{-4} m²/kN); m_1^s = 0.001 in.²/lb (1.45 × 10^{-4} m²/kN); R_a = −0.01; R_w = 0.5; and R_s = 0.5.
*Gauge.

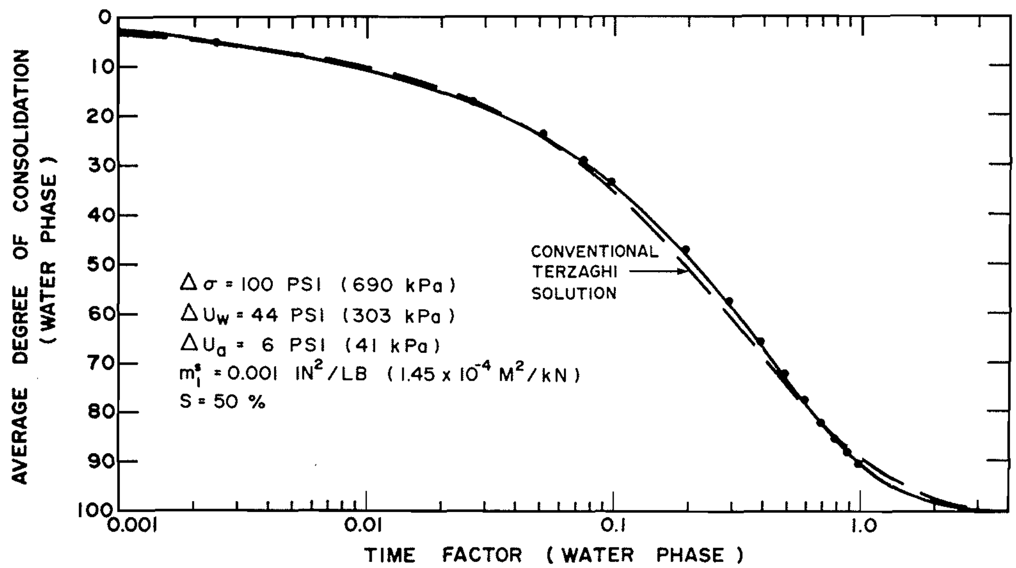

FIG. 7. Average degree of consolidation versus time factor.

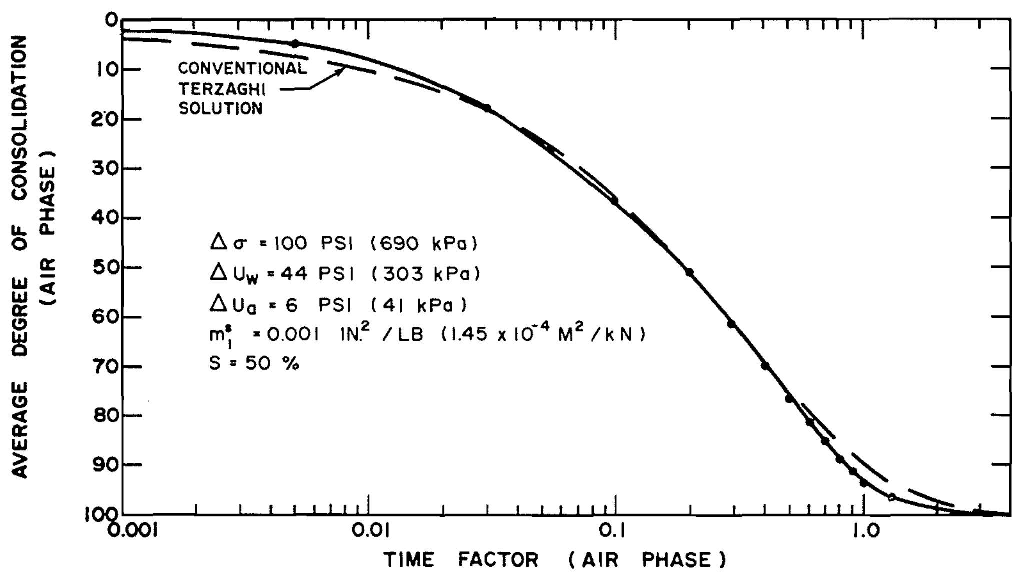

FIG. 8. Average degree of consolidation versus time factor.

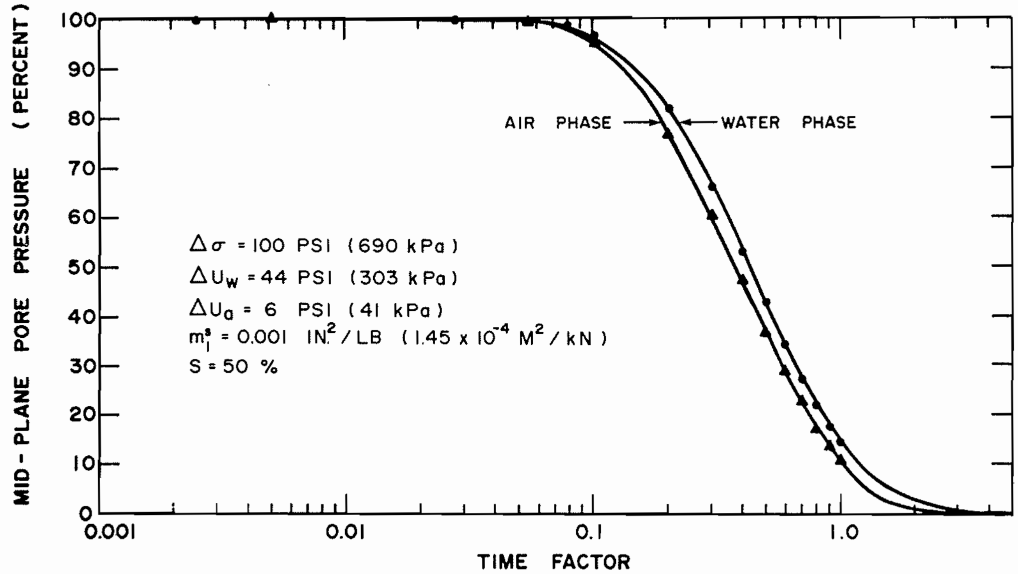

FIG. 9. Percent mid-plane pore-air and pore-water pressure versus time factor.

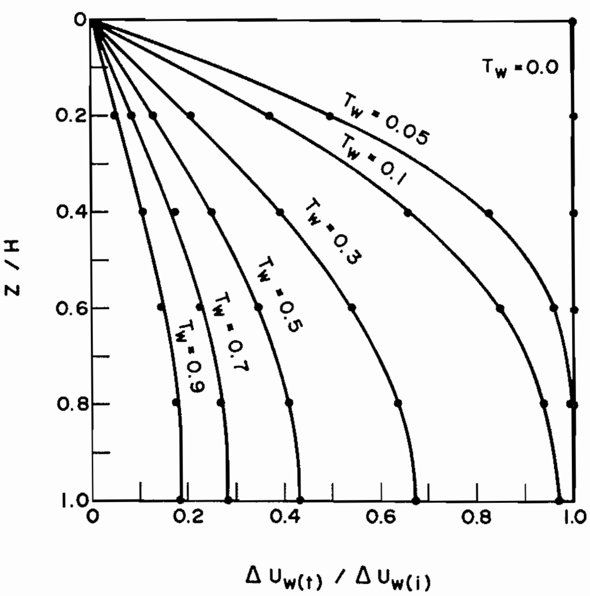

FIG. 10. Dimensionless isochrones for the water phase.

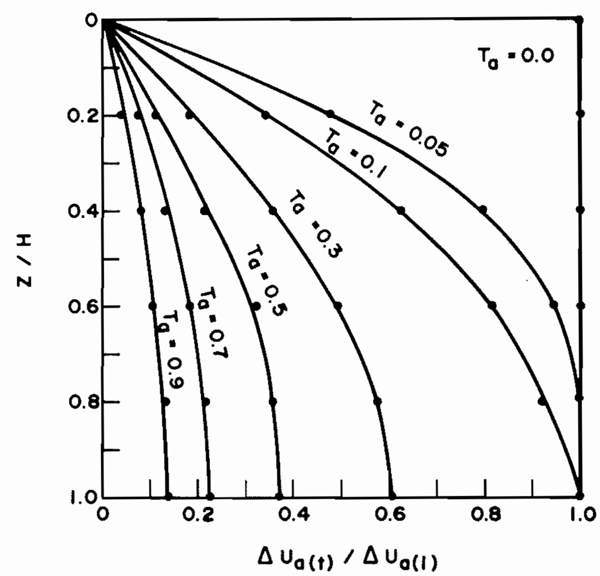

FIG. 11. Dimensionless isochrones for the air phase.

Fig. 8. It also shows a slight deviation from the Terzaghi solution for saturated soils. It should be noted that the c_v^a term has been assumed constant during the consolidation process. To do this, the absolute air pressure has been set to a constant equal to the average air pressure during the consolidation process. The error involved for this example problem is negligible.

Figure 9 shows the percent mid-plane pore-water and pore-air pressures plotted against the dimensionless time factors for both of the fluid phases. Dimensionless isochrones for both the water and the air phases are shown in Figs. 10 and 11 respectively. The plots show only slight deviations from the conventional Terzaghi plots. Other example problems have shown the interaction effects between the air and water phase partial differential equations to be more extreme.