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STRESS STATE VARIABLES FOR UNSATURATED SOILS

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INTRODUCTION

Practicing engineers are well aware that many of the problems they encounter involve unsaturated soils. The construction of earthfill dams, highways, and airport runways all use compacted soils that are unsaturated. Also, large portions of the earth's land surface are subjected to desiccating influences that leave the upper portion of the profile cracked and unsaturated. The success of the effective stress concept in describing the behavior of saturated soils has led research workers into a search for a similar statement for unsaturated soils. During the last two decades, there have been numerous equations proposed in the literature; however, none has proven completely successful in practice.

This paper examines the description of the stress state of an unsaturated soil within the context of multiphase continuum mechanics. First, the physical nature of an element of unsaturated soil is described and special consideration is given to the air-water interphase. Second, equilibrium equations for each phase of an unsaturated soil are written in terms of measurable variables. On the basis of these equations, it is possible to identify the independent variables necessary to describe the stress state in an unsaturated soil. Third, theoretically proposed stress state variables are then verified experimentally.

The next steps in the analysis of an unsaturated soil involve the prediction of the deformation state variables and the proposal of suitable constitutive relationships to combine the stress and deformation state variables (not dealt with in this paper). The constitutive equations can then be applied to the solution of practical engineering problems (18,20).

UNSATURATED SOIL ELEMENT

Generally, an unsaturated soil is considered to be a three-phase system. Lambe and Whitman (29) state that soil contains "three distinct phases; solid (mineral

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TABLE 1.—Effective Stress Equations for Unsaturated Soils

Reference number (1)	Equation (2)	Description of Variables (3)
5	$\sigma' = \sigma - u_a + \chi(u_a - u_w)$	χ = parameter related to degree of saturation u_a = the pressure in gas and vapor phase
12	$\sigma' = \sigma - \beta' u_w$	β' = holding or bonding factor which is measure of number of bonds under tension effective in contributing to soil strength
28	$\sigma = \bar{\sigma} a_m + u_a a_a + u_w a_w + R - A$	a_a = fraction of total area that is air-air contact
1	$\sigma' = \sigma + \psi p''$	ψ = parameter with values ranging from zero to one p'' = pore-water pressure deficiency
26	$\sigma' = \sigma + \beta p''$	β = statistical factor of same type as contact area; should be measured experimentally in each case
32	$\sigma' = \sigma - u_a + \chi_m(h_m + u_a) + \chi_s(h_s + u_a)$	χ_m = effective stress parameter for matrix suction h_m = matrix suction χ_s = effective stress parameter for solute suction h_s = solute suction

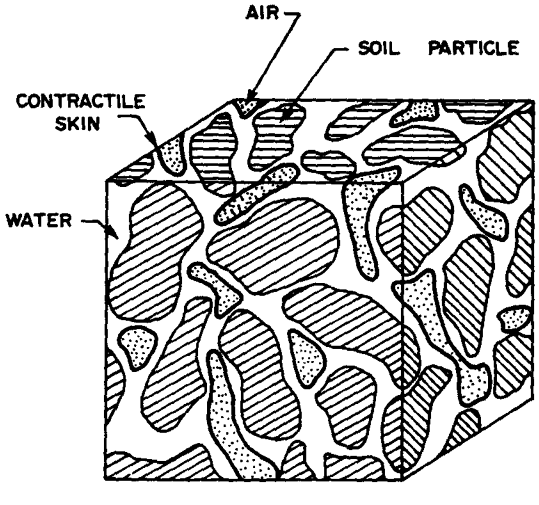


FIG. 1.—Element of Unsaturated Soil

particles); gas and liquid (usually water)." However, on the basis of the definition of a phase, the authors postulate that the air-water interface should be considered as a fourth and independent phase. The following characteristics (22,33) qualify a material as an independent phase of a mixture: (1) Differing properties than that of the surrounding material; and (2) definite bounding surfaces. Davies and Rideal clarify the nature of the air-water interface when they state (13):

> The boundary between two homogeneous phases is not to be regarded as a simple, geometrical plane, upon either side of which extends the homogeneous phases, but rather as a lamina or film of a characteristic thickness: the material in this surface phase shows properties differing from those of the materials in the contiguous homogeneous phases.

The air-water interface is commonly referred to as the "contractile skin" in the surface chemistry literature (31).

An element of an unsaturated soil can therefore be visualized as a mixture with two phases that come to equilibrium under applied stress gradients (i.e., soil particles and contractile skin) and two phases that flow under applied stress gradients (i.e., air and water) (Fig. 1). The terms "partly saturated" and "partially saturated" are generally used to designate the preceding case; however, the writers prefer the term "unsaturated," since the inclusion of the smallest amount of free gas renders the system unsaturated.

EFFECTIVE STRESS CONCEPT

The success of the effective stress equation for saturated soils (34) has led many researchers into a search for a similar equation for unsaturated soils (Table 1). The equations differ primarily in that some account for variations in air pressure and variations in solute suction. Common to all equations is the incorporation of a soil parameter characteristic of the soil behavior in the description of the stress state. This renders the equation a constitutive relationship rather than a description of the stress state (21). The soil parameter has proven essentially impossible to evaluate uniquely and difficult to apply to practical problems (6,8,9,10,11,27). More recently, there has been an increased tendency to uncouple the effective stress equations and treat the stress variables independently (2,3,30).

THEORETICAL EQUILIBRIUM ANALYSIS OF UNSATURATED SOIL ELEMENT

The procedure used herein to propose suitable stress state variables for an unsaturated soil is based on multiphase continuum mechanics. A three-dimensional element (i.e., a cube) is selected for the free body diagrams, of a size such that it contains a large number of particles and qualifies as a continuum (21). The equations of motion (or force equilibrium) are written for each phase of the element (i.e., water, air, soil particles, and contractile skin). In other words, it is assumed that each phase has an independent continuous stress field associated with each of the cartesian coordinate directions. The principle of superposition of coincident equilibrium stress fields is used in writing the equations of motion (15,23,35,36). In addition to the stress fields for each phase, an overall or total stress field can be assumed. However, the number of independent equations is equal to the number of phases involved. The derivations must be conducted such that all surface tractions are physically measurable quantities.

The derivations of the equilibrium equations for the soil particles and the contractile skin are presented in Appendix I. The surface tractions in each equation can be extracted and written in the form of stress matrices. Each surface traction is referred to as a "stress state variable." The extracted stress state variables cannot be placed in one matrix since they are linked by differing porosity terms outside the partial differentials. In any case, the inclusion of a porosity term (i.e., a soil property) in the description of the state of stress is not in keeping with continuum mechanics. Two independent stress matrices can be extracted from the equilibrium equations for the soil particles and the contractile skin. They are (Fig. 2):

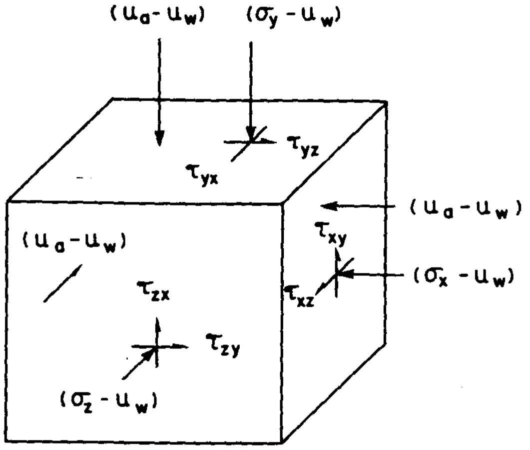


FIG. 2.—Stress State for Soil Particles and Contractile Skin in Unsaturated Soil

$$\begin{bmatrix} \sigma_x - u_w & \tau_{yx} & \tau_{zx} \\ \tau_{xy} & \sigma_y - u_w & \tau_{zy} \\ \tau_{xz} & \tau_{yz} & \sigma_z - u_w \end{bmatrix} \tag{1a}$$

and

$$\begin{bmatrix} u_a - u_w & 0 & 0 \\ 0 & u_a - u_w & 0 \\ 0 & 0 & u_a - u_w \end{bmatrix} \tag{1b}$$

The derivation in Appendix I used the water phase as a reference phase. However, if the air phase is used as a reference, the equilibrium equation is of a slightly different form. The first matrix will have u_a substituted for u_w. The total stress can also be used as a reference. In this case, the normal stress varibles are $(\sigma - u_a)$ and $(\sigma - u_w)$. Therefore, the analysis indicates that any two of three possible normal stress variables can be used to define the stress state. Possible combinations are: (1) $(\sigma - u_w)$ and $(u_a - u_w)$; (2) $(\sigma$

$- u_a$) and $(u_a - u_w)$; and (3) $(\sigma - u_a)$ and $(\sigma - u_w)$. Prior to their use in the formulation of constitutive relationships, they should be verified experimentally.

EXPERIMENTAL VERIFICATION OF STRESS STATE VARIABLES

It is now desirable to develop an experimental technique to verify the theoretically proposed stress state variables. The writers propose the following definition and proof for independent stress state variables:

> A suitable set of independent stress state variables are those that produce no distortion or volume change of an element when the individual components of the stress state variables are modified but the stress state variables themselves are kept constant. Thus the stress state variable for each phase should produce equilibrium in that phase when a stress point in space is considered.

Any proposed stress tensor can be considered with respect to the following question. How can the individual components of the state variables be altered without changing the overall stress matrix? The alteration of any one stress state variable should destroy the equilibrium of the system.

For a saturated soil, the test required to verify $(\sigma - u_w)$ as the stress state variable could be stated as follows: $\Delta\sigma_x = \Delta\sigma_y = \Delta\sigma_z = \Delta u_w$. If the deformation of the soil structure is zero during such a test, the proposed stress state variables are defined. In the case of an unsaturated soil, two stress matrices are proposed for both the soil particles and the contractile skin. For example, let the matrices be selected that contain the $(\sigma - u_w)$ and $(u_a - u_w)$ stress state variables.

To test the $(\sigma - u_w)$ matrix, the total stresses in all directions can be increased or decreased by an equal amount along with an equal increase or decrease in the water pressure. However, the change in the water pressure disturbs the equilibrium of the $(u_a - u_w)$ matrix. To maintain equilibrium in the $(u_a - u_w)$ matrix, the air pressure must also be increased or decreased by an amount equal to the water and total pressures. This test should satisfy the equilibrium of both the soil structure and the contractile skin. In other words, if all phases are operated as an open system (i.e., allowed to deform or flow if desired), there should be no tendency for volume change of the overall sample and no change in the degree of saturation. The test can be stated as $\Delta\sigma_x = \Delta\sigma_y = \Delta\sigma_z = \Delta u_w = \Delta u_a$. If the change in overall volume is zero and the change in degree of saturation is zero, equilibrium of the soil particles and contractile skin has been maintained.

These tests are termed null tests since the desired result of the changes in pressure is to not produce a process in the phase or phases under consideration. Thus, an attempt is being made to measure "no volume change" or a continuing equilibrium state. It is difficult to measure zero volume change over an extended period of time, and therefore slight volume changes would be anticipated. Similar null-type tests have been used in conjunction with shear strength tests on an unsaturated silt by Bishop and Donald (7). Also, the axis-translational technique for measuring matrix suction (24) is a special case of the null-type test where the total and air pressures are equal.

DESCRIPTION OF EQUIPMENT AND TEST PROGRAM

Four pieces of equipment were used in performing the null tests (16). Two employed modified Anteus oedometers (manufactured by Testlab Corporation in the United States) with one-dimensional loading conditions. The other two pieces of equipment allowed isotropic volume change testing conditions in

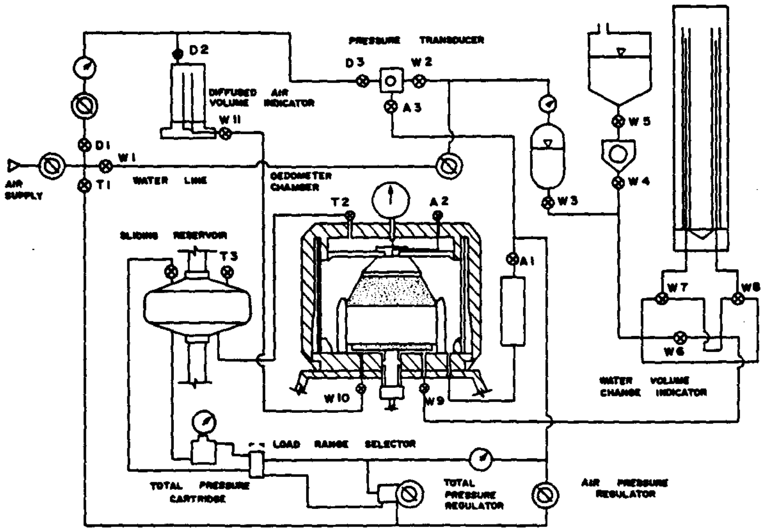


FIG. 3.—Layout of Modified Anteus Odometer

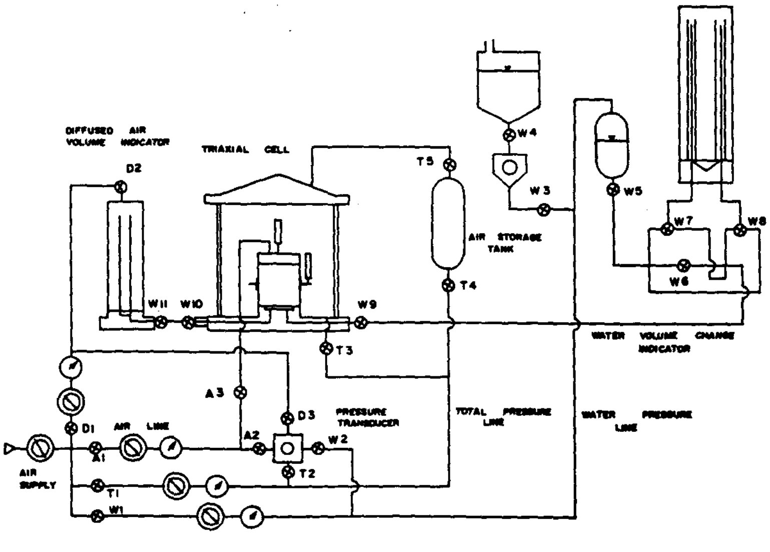


FIG. 4.—Layout of Modified Triaxial Apparatus

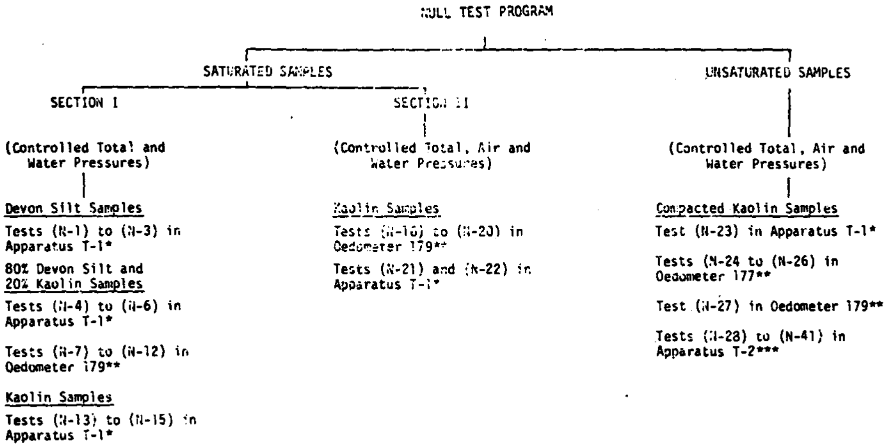
modified 4-in. diam Wykeham Farrance triaxial cells.

Air and water pressures were separated by means of a high air entry disk placed at the bottom of the sample [i.e., either 4 bar or 15 bar (400 kN/m^2 or 1,500 kN/m^2)]. Although the high air entry disks do not leak air at pressure less than their air entry value, dissolved air diffuses through the water in the disk and collects below the base of the disk. For accurate water volume change measurements, it was necessary to measure the volume of diffused air and apply the appropriate correction. The diffused air volume was measured by

TABLE 2.—Soil Properties of Samples Used for Null Tests

Soil type (1)	Samples tested (2)	Specific gravity (3)	Liquid limit, as a percentage (4)	Plastic limit, as a percentage (5)	Silt[a] sizes, as a percentage (6)	Clay[a] sizes, as a percentage (7)
Devon silt	1	2.695	31	23	80	18
80% Devon silt plus 20% Kaolin	2,3	2.679	38	25	67	31
Kaolin	4,6,7,9,10 11, 20, 22 27, 30, 31	2.616	64	35	15	85

[a] MIT Classification.



* Apparatus T-1 is a 4-inch diameter (102mm) double-walled triaxial cell where lateral displacements were monitored (using a cathetometer) by measuring fluctuations in the mercury level. Same as Bishop and Donald (7).

** Oedometers 177 and 179 were modified, one-dimensional Anteus Oedometer (15).

*** Apparatus T-2 is a 4-inch diameter (102mm) triaxial cell where LVDT's were used to measure lateral displacements (15).

FIG. 5.—Null Test Program

periodically flushing the base plate and measuring the volume of air displacing water in an inverted buret (19).

Fig. 3 shows the layout of the modified Anteus oedometer and associated plumbing and equipment. The chamber of the odometer, (normally filled with water for back pressuring the water phase of soils) was filled with air to regulate air pressure in the soil sample. All pressures were controlled using pressure regulators with an accuracy better than ±0.02 psi. The valves in Fig. 3 associated with the total, air, water, and diffused air pressure control are labeled with a T, A, W, and D, respectively.

The modified Wykeham Farrance triaxial cell (T-2) is shown in Fig. 4. Linear voltage displacement transducers (LVDT) were used to measure the vertical and lateral displacements. The cell was pressurized with air. A composite membrane consisting of slotted aluminum foil, vacuum grease, and two latex membranes was used to prevent the diffusion of air from the cell into the sample. The original Wykeham Farrance triaxial cell (T-1) had a double-walled cell with mercury between the sample and the inner cell. Fluctuations in mercury level were measured using a cathetometer. In the null test program, a series of tests was first performed on saturated samples, and then a series was performed on unsaturated samples. The properties of the soil tested are shown in Table 2. A summary of the test program is shown in Fig. 5.

ANALYSIS OF NULL TESTS

The key question considered during the analysis of the null tests is: Does any process or volume change occur as a result of the changes in pressure? If some volume change is measured, an attempt is made to ascertain its cause. Some of the main factors considered with respect to possible volume changes are: (1) Test procedure; (2) air diffusion through the water or membrane; (3) water loss from the sample; and (4) secondary consolidation.

The first 15 tests on the saturated samples (Section I) involved changing the total and water pressures and monitoring the water and overall sample volume changes. These tests served to assess the reliability of the test procedure and equipment for a known case. The results are presented in Table 3. Col. 5 shows the amount by which the total and water pressures were changed while Cols. 3 and 4 show the final total and water pressures, respectively. In Cols. 6, 7, and 8, a positive sign designates a volume decrease and vice versa. The "estimated volume change" in Col. 10 is the volume change that would have occurred if only the total stress had been changed and drainage allowed.

The first few tests showed that small amounts of entrapped air in the sample resulted in significant total and water volume changes. After both the testing technique and the means of removing the initially entrapped air were improved (i.e., vibration and a vacuum pump), the results were more satisfactory. The last set of null tests indicated a total or overall volume change of approx 1.8% of that estimated for a corresponding effective stress increase. The corresponding water volume change was approx 0.5%. These volume changes are not interpreted as a limitation of $(\sigma - u_w)$, being the effective stress variable, for a saturated soil but rather as an indication of testing technique limitations. The immediate total volume changes are related to entrapped air and measuring system compliance. Long-term volume changes are related to slight amounts of secondary

TABLE 3.—Summary of Null Tests on Saturated Samples, Controlling Total and Water Pressures

Test number (1)	Sample number (2)	After Pressure Change, in pounds per square inch		Pressure change, in pounds per square inch (5)	Sample Volume Change, as a percentage		Water volume change, as a percentage (8)	Elapsed time, in minutes (9)	Estimated volume change for corresponding effective stress increase, as a percentage (10)	Water content, as a percentage (11)	Void ratio (12)
		Total (3)	Water (4)		Immediate (6)	At elapsed time (7)					
N-1[a]	1	21.48	14.34	+10	—	—	−0.98	1,710	4.28	24.24	0.672
N-2[b]	1	31.41	24.87	+10	—	+0.72	−0.72	1,010	4.54	24.82	0.678
N-3[b]	1	41.72	1.47	−40	—	0.0	+2.37	1,000	1.14	24.02	0.620
N-4	2	40.11	32.23	+10	+0.08	+0.08	−0.06	1,000	5.02	28.53	0.783
					+0.08	+0.08	−0.11	6,000	5.02		
N-5	2	69.51	31.76	−20	—	+0.125	+0.02	2,000	2.77	24.32	0.655
N-6	2	44.66	6.93	−25	—	−0.22	+0.06	800	3.31	24.35	0.652
N-7[c]	3	74.11	60.18	+10	+0.011	+0.014	—	190	3.36	27.55	0.738
N-8[c]	3	83.71	69.78	+10	+0.021	+0.020	—	1,035	3.36	27.53	0.738
N-9[c]	3	73.92	59.99	−10	−0.003	−0.010	—	23	3.36	27.52	0.737
N-10[c]	3	63.92	49.99	−10	−0.010	−0.002	—	70	3.36	27.52	0.737
N-11[c]	3	53.50	39.57	−10	+0.003	+0.011	—	60	3.36	27.52	0.737
N-12[c]	3	43.85	29.92	−10	−0.003	+0.006	—	65	3.36	27.52	0.737
N-13	4	25.73	15.62	+10	−0.02	+0.10	+0.03	1,400	5.92	55.27	1.476
N-14	4	35.80	25.84	+10	+0.07	+0.10	+0.025	1,170	5.92	55.25	1.473
N-15	4	45.89	35.80	+10	+0.11	+0.11	0.0	120	—	55.22	1.470
					+0.11	+0.13	+0.012	1,470	5.92		

[a] Increased in 1-psi (6.89-kN/m²) increments.

[b] Increased in 5-psi (34.5-kN/m²) increments.

[c] Measured total volume change only.

TABLE 4.—Summary of Null Tests on Saturated Soils:

Test number (1)	Sample number (2)	After Pressure Change, in pounds per square inch			Pressure change, in pounds per square inch (6)	Sample Vol as a per
		Total (3)	Air (4)	Water (5)		Immediate (7)
N-16	6	56.40	39.27	30.08	+10	+0.054
						−0.073
N-17	6	66.53	49.36	40.33	+10	−0.073
						−0.073
N-18	6	56.25	39.19	30.44	−10	0.0
N-19	6	66.23	49.14	40.50	+10	+0.006
						+0.006
N-20	6	76.28	59.19	50.08	+10	+0.012
N-21	7	74.04	54.43	53.99	+8	0.0
N-22	7	83.90	64.22	44.06	+10	0.0

Note: 1 psi = 6.89 kN/m^2.

consolidation of the soil structure and compression of the pore air. The results are sufficiently accurate to verify the stress state variables for engineering purposes.

In the tests performed for Section II, the top of the saturated samples was subjected to an air pressure change in addition to the application of the total and water pressure changes (Table 4). The total and water volume changes were monitored on all tests. Air pressure was applied through a coarse corundum stone to create a differential between the pore-water pressure and the air pressure applied to the surface of the sample. Some difficulty occurred in applying all three pressures simultaneously. In the one-dimensional oedometer, the air pressure took a few seconds to increase. As a result, the sample underwent a slight immediate compression. A second problem was the diffusion of air

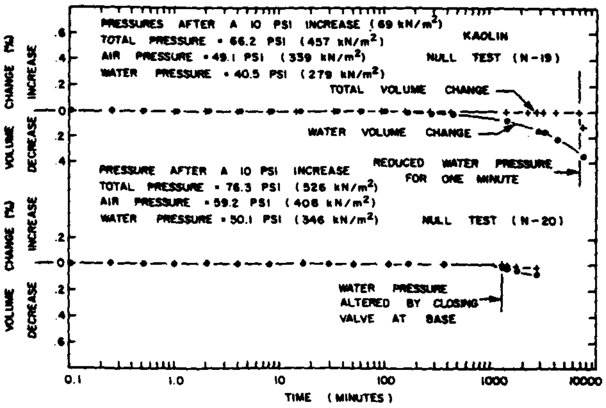


FIG. 6.—Null Tests (N-19) and (N-20) on Sample No. 6

Controlling Total, Air, and Water Pressures

ume Change, centage	Water volume change, as a percentage	Elapsed time, in minutes	Estimated volume change for corresponding effective stress increase, as a percentage	Water content, as a percentage	Void ratio
At elapsed time (8)	(9)	(10)	(11)	(12)	(13)
+0.063	+0.168	1,410	2.93	49.91	1.309
−0.066	+0.054	100	—		
−0.069	+0.19	1,000	2.93	49.81	1.307
−0.062	+0.54	4,300	2.93		
+0.019	+0.28	2,760	2.93	49.21	1.309
0.00	+0.05	1,000	2.93	48.91	1.308
−0.001	+0.33	7,180	2.93		
+0.027	+0.088	2,770	2.93	48.56	1.305
+0.129	−0.057	2,870	3.07	50.37	1.309
+0.110	+0.148	7,170	2.14	44.95	1.203

TABLE 5.—Summary of Null Tests on Unsaturated Soils: Controlling Total, Air and Water Pressures

Test number (1)	Sample number (2)	After Pressure Change, in pounds per square inch			Stress change, in pounds per square inch (6)	Water content, as a percentage (7)	Void ratio (8)
		Total (3)	Air (4)	Water (5)			
N-23	9	71.37	50.62	26.15	+10	31.56	1.087
N-24	10	71.84	59.00	20.81	+20	31.16	0.870
N-25	10	81.78	68.90	30.51	+10	31.21	0.870
N-26	10	72.14	59.30	42.56	−30	31.73	0.873
N-27	11	43.95	30.00	26.27	+10	32.20	0.902
N-28	20	88.67	77.16	24.68	+20	32.03	0.928
N-29	22	49.76	39.23	13.23	+10	34.59	1.062
N-30	22	59.74	49.17	23.34	+10	35.10	1.062
N-31	22	69.54	58.93	33.00	+10	35.82	1.028
N-32	22	79.62	69.09	43.11	+10	35.87	1.025
N-33	22	89.63	78.96	53.02	+10	35.90	1.022
N-34	27	49.27	38.88	20.19	+10	34.75	1.091
N-35	31	69.68	59.14	40.32	+10	34.01	1.026
N-36	31	79.41	68.70	49.88	+10	34.05	1.025
N-37	31	89.25	78.50	59.65	+10	34.07	1.026
N-38	31	79.69	69.19	50.41	−10	34.11	1.025
N-39	31	69.50	59.11	40.29	−10	34.10	1.025
N-40	31	59.84	49.41	30.66	−10	34.11	1.025
N-41	31	39.47	29.07	10.39	−20	34.09	1.025

Note: 1 psi = 6.89 kN/m^2.

through the water phase of the sample. This was reflected as water leaving the sample. However, the total or overall volume of the sample remained essentially constant (Fig. 6). The best test results show the total volume change over a period of several days as less than 0.1% for a 10-psi (69-kN/m^2) pressure increase on a soft saturated soil. This is approx 2% of that anticipated for a corresponding change in effective stress.

A total of 19 null tests were performed on unsaturated samples (Tables 5 and 6). Samples N-23 to N-28 were prepared at standard American Association of State Highway and Transportation Officials (AASHTO) compactive effort whereas the remaining samples were prepared at one-half standard AASHTO. The first few tests performed showed remarkably low volume changes. There was a slight tendency for some water to go into the sample during the early stages (i.e., up to 100 min) of the test, but later there was evidence of a small volume of water leaving the sample. The phenomenon is readily understood when it is realized that, at the moment the pressures are applied, two secondary transient processes commence. Air starts to diffuse through the water primarily from the top of the sample while water tends to compress the occluded air bubbles from the bottom of the sample. The entrapped air bubbles appear to

TABLE 6.—Sample Volume Change and Water Volume Change During Null Tests on Unsaturated Soils: Controlling Total, Air, and Water Pressures

Test number (1)	Sample Volume Change, as a percentage		Water volume change, as a percentage (4)	Elapsed time, in minutes (5)	Estimated Volume Change for Corresponding Effective Stress Change, as a percentage	
	Immediate (2)	At elapsed time (3)			Overall sample (6)	Water phase (7)
N-23	0.0	−0.03	−0.05	5,800	0.6	2.3
N-24	+0.04	+0.4	−0.07	1,500	1.4	0.8
N-25	+0.01	0.0	−0.02	1,650	0.7	0.4
N-26	−0.25	−0.20	—	4,300	2.1	1.2
N-27	0.0	−0.10	−0.50	1,880	0.8	0.5
N-28	≈−0.15	−0.15	−0.11	1,900	1.6	1.8
N-29	≈−0.015	+0.012	−0.642	8,700	1.0	1.2
N-30	≈−0.005	+0.012	−0.072	1,350	1.0	1.2
N-31	—	+0.12	−0.060	1,380	1.1	0.8
N-32	—	+0.17	−0.045	1,390	1.1	0.8
N-33	—	+0.15	−0.020	410	1.1	0.8
N-34	≈+0.055	+0.060	−0.105	4,350	1.0	2.5
N-35	+0.015	+0.033	−0.060	5,800	0.9	1.9
N-36	+0.010	−0.020	−0.035	2,800	0.9	1.9
N-37	0.0	−0.005	−0.050	5,800	0.9	1.9
N-38	−0.015	+0.002	+0.010	2,700	0.9	1.9
N-39	−0.010	+0.005	−0.005	1,500	0.9	1.9
N-40	−0.007	−0.005	+0.015	5,800	0.9	1.9
N-41	−0.030	+0.007	−0.040	2,900	1.8	3.8

equalize with time, whereas the diffusion process tends toward a steady-state process.

Subsequent to the initial tests, a new triaxial apparatus (T-2) was designed with more complete data acquisition capabilities. The null test program was used to simultaneously assess the performance of the apparatus and perform further precise null tests. Two new problems were encountered. First, there was a delay in the application of the cell pressure which allowed the upper load cap to be lifted from the sample at the start of the test. Second, there was a slow leakage of water through the rubber membranes surrounding the sample. The first problem was overcome by using a surge tank to rapidly increase cell pressure. The second problem was overcome by means of a composite membrane consisting of two rubber membranes separated by slotted aluminium foil and vacuum grease (14).

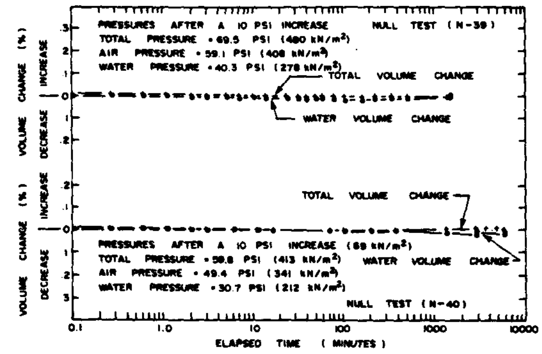


FIG. 7.—Null Tests (N-39) and (N-40) on Sample No. 31

FIG. 8.—Null Test (N-37) on Unsaturated Sample No. 31 (Expanded Scale)

The last set of null tests (i.e., N-35 to N-41) were performed on Triaxial Apparatus No. 2 with the procedural problems solved. Fig. 7 shows the results of null tests N-39 and N-40. The results show essentially no volume change in either the total sample or the water phase. Expanding the volume change scale by 10 times on null test N-37 (Fig. 8) shows that the results are approaching the accuracy of the measuring systems. The volume changes after 1 day are within one part in 10,000. The total volume changes were approx 0.3% of the volume change that would have occurred if only the total stress had been changed. The water volume changes were approx 2.5% of those that would have occurred if only the water pressure had been changed. Even when taking into account the increased stiffness of the samples, the two stress-state variables for the unsaturated soils are verified at least as conclusively as the effective stress variable was for the saturated samples.

These results are encouraging and supply evidence to state with assurance that the theoretically proposed stress state variables qualify as satisfactory stress state variables from an experimental standpoint.

CONCLUSIONS

Previous experimental evidence substantiates the consideration of the air-water interface (i.e., contractile skin) as an independent phase. It qualifies as an independent phase due to its distinct physical properties. Therefore, an unsaturated soil is a four-phase system, composed of two phases that come to equilibrium under applied stress gradients, (i.e., soil particles and contractile skin) and two phases that flow under applied pressure gradients (i.e., air and water phases).

Force equilibrium equations for each phase of an unsaturated soil are formulated within the context of multiphase continuum mechanics. They are arranged in such a manner that the stresses associated with each phase are written in terms of the physically measurable stresses (i.e., σ, u_a, and u_w). The stress state variables governing the behavior of an unsaturated soil (i.e., soil particles and contractile skin) are extracted from the equilibrium equations to form two independent stress matrices. The analysis indicates that any two of three possible normal stress variables can be used to define the stress state. Possible combinations are: (1) $(\sigma - u_w)$ and $(u_a - u_w)$; (2) $(\sigma - u_a)$ and $(u_a - u_w)$; and (3) $(\sigma - u_a)$ and $(\sigma - u_w)$.

Null tests were performed to experimentally test the proposed stress-state variables. Experimental data from numerous tests on unsaturated soil samples indicated essentially no overall volume change or water volume change during the null tests. On the final set of tests, the overall and water volume changes were approx 2% of the volume changes that would be associated with a change in either one of the stress state variables. The theoretically proposed stress state variables would appear to be verified for the soil structure and the contractile skin.

APPENDIX I.—EQUILIBRIUM ANALYSIS OF UNSATURATED SOIL

The element most suitable for the equilibrium analysis of a multiphase system is a cube, completely enclosed by imaginary unbiased boundaries (4,25). However, many research workers in the area of soil mechanics have used statical equilibrium across a wavy cross-sectional plane passed through the soil when justifying the use of a proposed effective stress equation (6,29). The wavy plane constitutes a special type of free body diagram for which the spatial variation goes to zero. This is an equivalence statement in which one force system is substituted for another equivalent force system.

In this paper, the equilibrium of a cubical element is considered within the context of continuum mechanics applied to multiphase systems. An element of unsaturated soil can be disassembled into a water, air, soil particle, and contractile skin phase. Linear equilibrium equations can be written for each of the four phases (17). In addition, an overall or total equilibrium equation can also be written (Fig. 9) as

$$\left(\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} + \gamma\right) dx\,dy\,dz = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

in which γ = total unit weight of saturated soil; σ_y = total stress in y direction; τ_{xy} = shear stress on x-plane in y direction; τ_{zy} = shear stress on z-plane in y direction; and dx, dy, dz = unit dimensions of element.

This gives rise to five equilibrium equations in each of the cartesian coordinate directions. However, only four equations are independent. Since it is possible to measure only total, pore-water, and pore-air pressure, the equilibrium equations for the soil particles and contractile skin will be written in terms of the total, water, and air phase equilibrium equations.

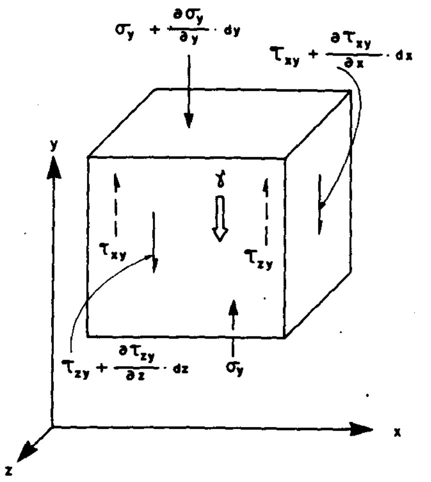


FIG. 9.—Y-Direction Equilibrium for Overall Element of Unsaturated Soil

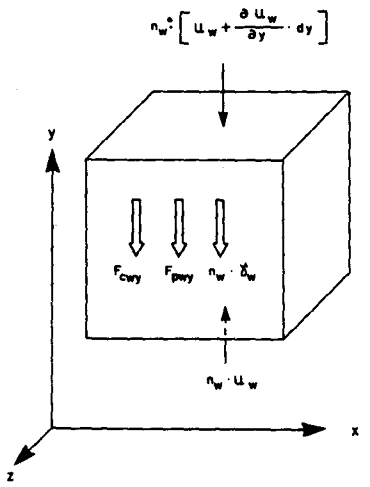


FIG. 10.—Y-Direction Equilibrium for Pore-Water Phase of Unsaturated Soil

Fig. 10 shows the forces in the y direction associated with the water phase. Assuming that the contractile skin behaves as a membrane that comes to equilibrium, there is an interaction force between the water and the contractile skin which can be represented by a body force, F_{cw}. There is also an interaction force between the water and the soil particles, F_{pw}. The equilibrium in the y direction is

$$\left(n_w \frac{\partial u_w}{\partial y} + n_w \gamma + F_{cwy} + F_{pwy}\right) dx\,dy\,dz = 0 \quad \ldots\ldots\ldots\ldots\ldots (3)$$

in which u_w = pressure in water phase; n_w = porosity with respect to water phase; and $n_w \gamma_w$ = gravity body force for water phase.

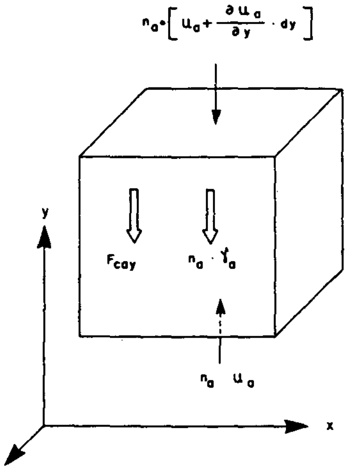


FIG. 11.—*Y*-Direction Equilibrium for Pore-Air Phase of Unsaturated Soil

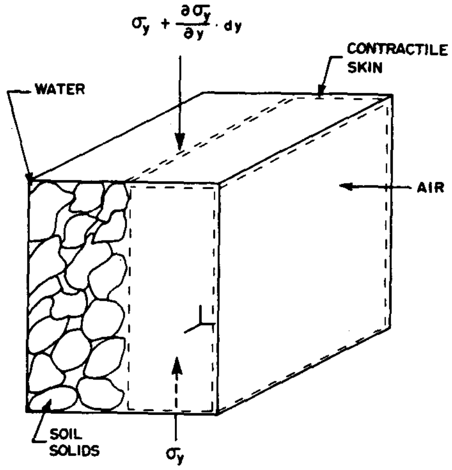


FIG. 12.—Rearrangement of Unsaturated Soil Element

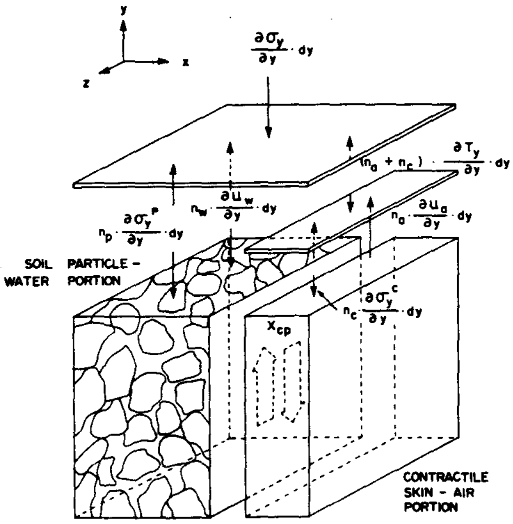


FIG. 13.—Rearranged Unsaturated Soil Element Separated into Two Portions

Fig. 11 shows the forces associated with the air phase. Let the interaction force between the air and the contractile skin be designated by a body force, F_{ca}. Summing in the y direction gives

$$\left(n_a \frac{\partial u_a}{\partial y} + n_a \gamma_a + F_{cay}\right) dx\, dy\, dz = 0 \quad \text{. . . . . (4)}$$

in which u_a = pressure in the air phase; n_a = porosity with respect to the air phase (i.e., percentage of the surface of the element going through air); and $n_a \gamma_a$ = gravity body force for the air phase.

The contractile skin is only a few molecular layers thick but can be described by an independent equilibrium equation since it qualifies as an independent phase and affects soil behavior. The contractile skin interacts with the soil particles producing an interaction force, X_{pc}. The drag of water and air on the contractile skin are F_{wc} and F_{ac}, respectively. Similarly, the soil particles can be described by an independent equilibrium equation. There is an interaction force, X_{cp}, between the contractile skin and the soil particles. Also, there is the drag force of the water on the soil particles, F_{wp}.

Although the equilibrium equations for the contractile skin and the soil particles can be written assuming a general state of stress, the associated stresses cannot be physically measured. This difficulty is resolved by uncoupling the unsaturated soil element into a water-soil particle multiphase and an air-contractile skin multiphase. The fictitious element has all the air moved to one side of the element (Fig. 12), with the total and component stress fields remaining the same. (The rearrangement of the element is strictly for conceptual purposes.) Now let the unsaturated soil element be separated into two portions. The first portion contains only soil particles and water but has additional forces applied to make it equivalent to the original system (Fig. 13). The stress field, $\partial T_y/\partial y$, accounts for the net normal effect of the air and contractile skin stress fields. Similar substitutionary stress fields must be set up on the sides of the element to replace the shear components. (These are omitted from Fig. 13 and the analysis in order to reduce the number of terms involved. They cancel out in the final equations.)

The body force, X_{cp}, must be applied to account for the interaction between the contractile skin and the soil particles. In addition, a unit weight term, γ_{ac}, must account for the air and contractile skin gravity force. The y direction equilibrium equation for the soil particles can be written as

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial(\sigma_y - u_w)}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} + (\gamma - n_w \gamma_w) + (n_p + n_a + n_c)\frac{\partial u_w}{\partial y}$$

$$- F_{cwy} - F_{pwy} - X_{cp} + (n_a + n_c)\frac{\partial T_y}{\partial y} + \gamma_{ac} = 0 \quad \text{. . . . . (5)}$$

in which n_p = percentage of element surface that is soil particles; and n_c = percentage of element surface that is contractile skin.

Using the air-contractile skin portion of the element (Fig. 13), the equilibrium equation for the contractile skin is

$$X_{cp} + (n_a + n_c)\frac{\partial T_y}{\partial y} + \gamma_{ac} - n_a \frac{\partial u_a}{\partial y} - n_a \gamma_a - F_{cay} = 0 \quad \text{. . . . . (6)}$$

Substituting Eq. 6 into Eq. 5 gives

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial(\sigma_y - u_w)}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} + (\gamma - n_w\gamma_w - n_a\gamma_a)$$

$$+ (n_p + n_c)\frac{\partial u_w}{\partial y} - F_{cwy} - F_{pwy} - F_{cay} - n_a\frac{\partial(u_a - u_w)}{\partial y} = 0 \quad \ldots\ldots\ldots (7)$$

This equilibrium equation applies for both the soil particles and the contractile skin. Similar equilibrium equations can be written for the x and z directions.

An examination of the equilibrium equation reveals that it contains three independent sets of surface tractions [i.e., $(\sigma - u_w)$, $(u_a - u_w)$, (u_w)]. The u_w term can be eliminated when the soil particles are assumed incompressible. The equilibrium equation for the soil particles in the case of a saturated soil can be similarly derived by subtracting the water-phase equation from the total stress equilibrium equation:

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial(\sigma_y - u_w)}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} + \gamma - n_w\gamma_w - F_{pwy} + n_p\frac{\partial u_w}{\partial y} = 0 \quad \ldots\ldots\ldots (8)$$

The normal stress components in the equilibrium equation are equivalent to the conventionally defined effective stress; i.e., the so-called effective stress law is actually a stress-state variable extracted from an equilibrium equation.

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12919 STRESS VARIABLES FOR UNSATURATED SOILS

KEY WORDS: Air-water interfaces; Continuum mechanics; **Effective stress; Saturated soils; Saturation; Saturation zones;** Stress analysis

ABSTRACT: An unsaturated soil is visualized as a four-phase system, the fourth phase being the air-water interface commonly referred to as the contractile skin. Suitable stress state variables for an unsaturated soil are proposed on the basis of writing force equilibrium equations for each phase, within the context of multiphase continuum mechanics. The analysis indicates that any two of three possible normal stress variables can be used to define the stress state. Experimental null-type tests verified the proposed stress state variables for the soil structure and contractile skin of an unsaturated soil.

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