CONSOLIDATION THEORY FOR UNSATURATED SOILS

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ABSTRACT

A one-dimensional consolidation theory is presented for unsaturated soils. The assumptions made are in keeping with those used in the conventional theory of consolidation for saturated soils, with the addition of the assumption that the air phase is continuous. Two partial differential equations are derived to describe the transient processes taking place as a result of the application of a total load to an unsaturated soil.

After a load has been applied to the soil, air and water flow simultaneously from the soil until equilibrium conditions are achieved. The simultaneous solution of the two partial differential equations gives the pore air and pore water pressures at any time and any depth throughout the soil.

For the case of an applied total load, two equations are derived which predict the initial pore air and pore water pressure boundary conditions. An example problem demonstrates the nature of the results.

INTRODUCTION

The practice of soil mechanics took on a theoretical context in the 1930's by Terzaghi's (12) presentation of the concept of effective stress and its application to the consolidation of saturated soils. Unfortunately, the study of the behavior of unsaturated soils has taken place in the absence of a similar theoretical framework. As a result, it has been difficult to envisage the transitions in theory when going from a saturated soil to an unsaturated soil. This paper presents a theory of consolidation (or swelling) for unsaturated soils within a theoretical framework similar to that used for saturated soils.

As far back as 1941, Biot (3) presented an analysis of the transient flow problem in unsaturated soils. He suggested two constitutive relations for the soil and solved for changes in the pore water pressure with time. Biot, considered the air to be in an occluded state with no flow of air during the consolidation process. Scott (11) showed a formulation for transient flow in an unsaturated soil which involves the use of only one constitutive relationship for the soil and does not consider flow associated with the air phase.

In 1965, Barden (2) presented an analysis of the one-dimensional consolidation of compacted, unsaturated clay. He subdivides the consolidation problem into various categories depending upon the degree of saturation of the soil. He states the problem is indeterminate and therefore, various assumptions are made in order to complete the analysis. Bishop's equation is used to describe the stress conditions in the unsaturated soil.

Partial differential equations have been developed in the soil physics discipline to describe unsteady moisture movement. More recently these equations have received increasing acceptance in the soil mechanics field (1). These equations should be considered as a special case since the compressibility of the soil structure and the escape of air are not taken into consideration. They are generally not applied to transient processes associated with the application of an external load.

Fredlund and Morgenstern (9) proposed stress state variables for unsaturated soils on the basis of the equilibrium equations for a multiphase system. These were also verified experimentally. The element of unsaturated soil was considered as a four phase system with the two phases that come to equilibrium under applied stress gradients (ie. soil particles and contractile skin or air-water interface) and two phases that flow under applied stress gradients (ie. air and water). Fredlund and Morgenstern (8) also proposed and experimentally tested constitutive relations for volume change in unsaturated soils. In addition, the continuity requirements for an element of unsaturated soil were outlined. The description of the stress, continuity and constitutive relations, along with suitable flow and compressibility laws for air and water, provide the necessary physical requirements for a more rigorous formulation of transient processes in unsaturated soils.

This paper presents a general one-dimensional consolidation (or swelling) theory for unsaturated soils (10). As well, equations are derived to predict the initial air and water boundary conditions associated with the application of an externally applied load. An example problem is included to demonstrate the solution of the above equations.

PHYSICAL REQUIREMENTS FOR THE FORMULATION

The state of stress in an unsaturated soil can be described by any two of a possible three stress state variables (9). Acceptable combinations are: (1. $(\sigma-u_a)$ and (u_a-u_w); (2. $(\sigma-u_w)$ and (u_a-u_w); (3. $(\sigma-u_a)$ and $(\sigma-u_w)$. The stress variables selected to derive the consolidation equations in this paper are: $(\sigma-u_a)$ and (u_a-u_w),

where σ = total stress

u_a = pore air pressure

u_w = pore water pressure

Continuity of an unsaturated soil element requires that the overall volume change of the element must equal the sum of the volume changes associated with the component phases (6). If the soil particles are considered incompressible and the volume change of the contractile skin (ie. air-water interface) is considered as internal to the element, the continuity requirement can be written,

$$[1] \qquad \frac{\Delta V}{V} = \frac{\Delta V_w}{V} + \frac{\Delta V_a}{V}$$

where V = overall volume of the soil element

V_w = volume of water in the soil element

V_a = volume of air in the soil element

Fredlund and Morgenstern (8) proposed and tested constitutive relations to link the stress and deformation state variables. The proposed constitutive relationship for the soil structure is given by equation [2] and the relationship for the water phase is given by equation [3].

$$[2] \qquad \frac{\Delta V}{V} = m_1^s \, d(\sigma - u_a) + m_2^s \, d(u_a - u_w)$$

$$[3] \qquad \frac{\Delta V_w}{V} = m_1^w \, d(\sigma - u_a) + m_2^w \, d(u_a - u_w)$$

where m_1^s = compressibility of the soil structure when $d(\sigma-u_a)$ is zero

m_2^s = compressibility of the soil structure when $d(u_a-u_w)$ is zero

m_1^w = slope of the $(\sigma-u_a)$ plot when $d(u_a-u_w)$ is zero

m_2^w = slope of the (u_a-u_w) plot when $d(\sigma-u_a)$ is zero

The constitutive relationship for the air phase is the difference between equations [2] and [3] because of the continuity requirement.

$$[4] \qquad \frac{\Delta V_a}{V} = m_1^a \, d(\sigma - u_a) + m_2^a \, d(u_a - u_w)$$

where m_1^a = slope of the $(\sigma-u_a)$ plot when $d(u_a-u_w)$ is zero

m_2^w = slope of the (u_a-u_w) plot when $d(\sigma-u_a)$ is zero

Flow of the water phase is described by Darcy's law (5).

$$[5] \qquad v = \frac{-k_w}{\gamma_w} \frac{\partial u_w}{\partial y}$$

where v = flow rate of water

k_w = coefficient of permeability with respect to the water phase

γ_w = density of water

y = depth in the y-direction

Flow of the air phase is described by Fick's law (4) as,

$$[6] \qquad v = -D \frac{\partial p}{\partial y}$$

where v = mass rate of air flow

D = a transmission constant having the same units as coefficient of permeability

p = absolute air pressure (ie. $u_a + u_{atm}$)

u_{atm} = atmospheric air pressure

Fredlund (7) presented the isothermal compressibility equation for the air phase, β_a, as,

$$[7] \qquad \beta_a = \frac{1}{(u_a+u_{atm})}$$

and the isothermal compressibility equation of an air-water mixture (with no diffusion) in the presence of a particulate mass, β_m, as,

$$[8] \qquad \beta_m = S\ \beta_w + B_{aw} \frac{(1-S)}{(u_a+u_{atm})}$$

where S = initial degree of saturation

B_{aw} = pore pressure coefficient equal to $\Delta u_a/\Delta u_w$.

The above listed physical relationships are sufficient to derive the one-dimensional consolidation (or swelling) equations and the boundary condition equations for an unsaturated soil.

DERIVATION OF THE CONSOLIDATION EQUATIONS

The one-dimensional consolidation equations for unsaturated soils are derived using the conventional assumptions for Terzaghi's consolidation theory with the following additions:

i) The air phase is continuous

ii) The coefficients of permeability with respect to water and air, and the volume change moduli remain constant during the transient processes.

iii) The effects of air diffusing through water and the movement of water vapor are ignored.

The above assumptions are not completely accurate for all cases; however, they are reasonable for a first attempt to derive a general consolidation theory for unsaturated soils.

A partial differential equation is derived for each of the two phases that flow, based on continuity with respect to each phase. Then the two equations are solved simultaneously to give the water and air pressures at any elapsed time.

Water Phase Partial Differential Equation

Let us consider a referential soil element as shown in Figure 1. The water phase is assumed incompressible. For the consolidation process, water flows out of the element with time. The constitutive relationship for the water phase defines the volume of water in the element for any combination of total, air and water pressures. The volume of water entering the element in the y-direction is written according to Darcy's law as,

$$[9] \qquad \text{Volume entering} = \frac{-k_w}{\gamma_w} \frac{\partial u_w}{\partial y} \, dx \, dz$$

The net flux of water in the element is,

$$[10] \qquad \frac{\partial (V_w/V)}{\partial t} = \frac{-k_w}{\gamma_w} \frac{\partial^2 u_w}{\partial y^2}$$

Equation [10] can be equated to the constitutive relationship for the water phase.

$$[11] \qquad m_1^w \frac{\partial(\sigma - u_a)}{\partial t} + m_2^w \frac{\partial(u_a - u_w)}{\partial t} = \frac{-k_w}{\gamma_w} \frac{\partial^2 u_w}{\partial y^2}$$

For the consolidation process the change in total stress with respect to time is set to zero. Simplifying and rearranging equation [11], the water phase partial differential equation can be written,

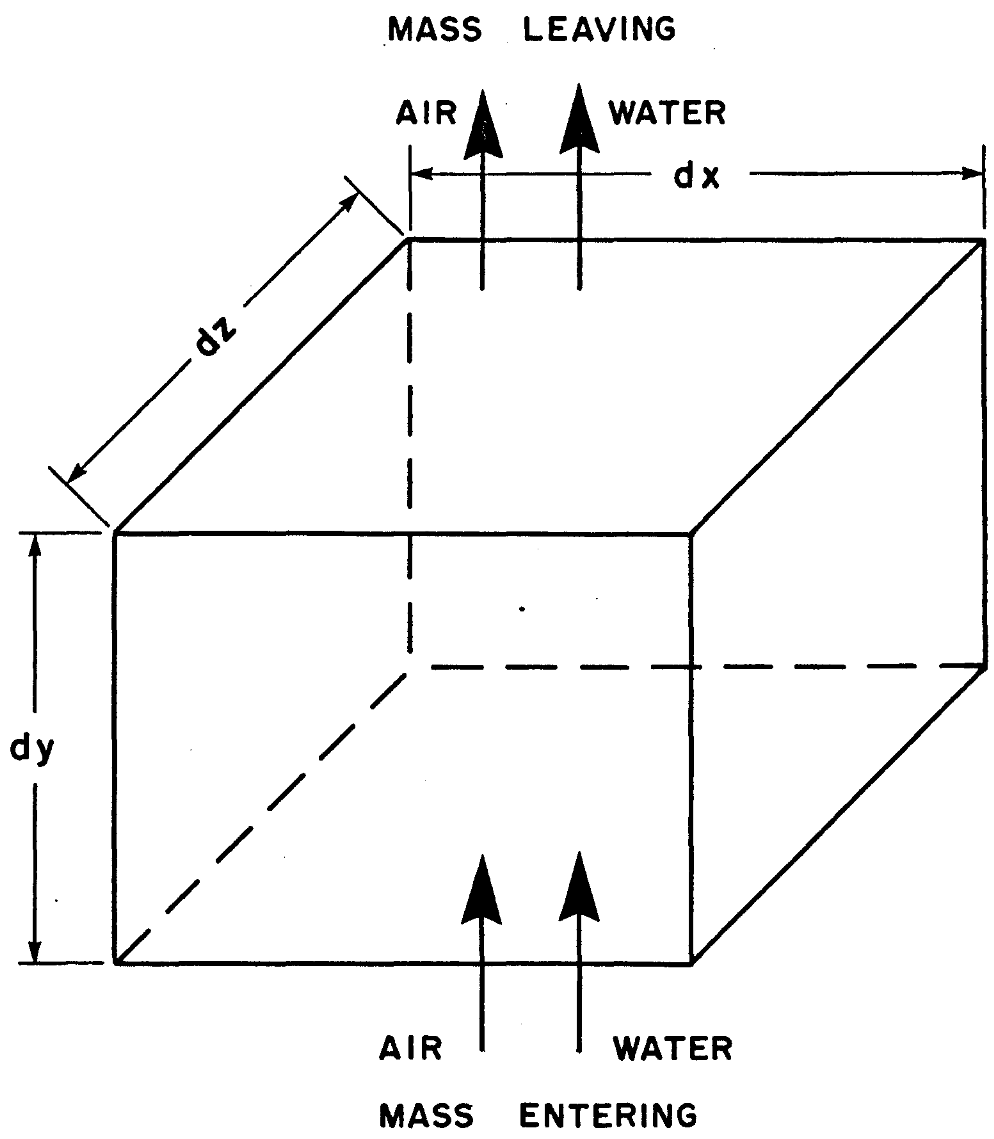

FIGURE I A REFERENTIAL ELEMENT IN THE SOIL MASS

$$[12] \qquad \frac{\partial u_w}{\partial t} = -C_w \frac{\partial u_a}{\partial t} + c_v^w \frac{\partial^2 u_w}{\partial y^2}$$

where $C_w = (1-m_2^w/m_1^w)/(m_2^w/m_1^w)$ and is called the interactive constant associated with the water phase equation. This equation is further simplified by defining R_w as m_2^w/m_1^w. When the soil is saturated, $R_w = 1$.

$c_v^w = \frac{1}{R_w} \frac{k_w}{\gamma_w \, m_1^w}$ and is called the coefficient of consolidation with respect to the water phase.

Air Phase Partial Differential Equation

The air phase is compressible and flows independent of the water phase when subjected to an air pressure gradient. As well, the constitutive relationship for the air phase defines the volume of air in the element for any combination of the total, air and water pressures. According to Fick's law the mass of air entering the element in the y-direction is,

$$[13] \qquad \text{mass entering} = -D \frac{\partial p}{\partial y} \, dx \, dz$$

The net mass flux of air in the element is,

$$[14] \qquad \frac{\partial m}{\partial t} = -D \frac{\partial^2 p}{\partial y^2}$$

where m = mass of air in the element

The mass rate of change is written in terms of a volume rate of change by differentiating the relationship between mass and volume.

$$[15] \qquad \frac{\partial(V_a/V)}{\partial t} = \frac{\partial(m/\gamma_a)}{\partial t}$$

For isothermal conditions the density of air, γ_a, is,

$$[16] \qquad \gamma_a = \frac{w}{R\theta}\, p$$

where w = molecular weight of the mass if air

R = universal gas constant

θ = absolute temperature

The mass of air is written in terms of the density of air, the degree of saturation, S, and the porosity of the soil, n.

$$[17] \qquad m = (1-S)n\,\gamma_a$$

Substituting equations [14], [16] and [17] into equation [15] gives,

$$[18] \qquad \frac{\partial(V_a/V)}{\partial t} = \frac{-DR\theta}{w\,p}\frac{\partial^2 u_a}{\partial y^2} + \frac{(1-S)n}{p}\frac{\partial u_a}{\partial t}$$

Equating equation [18] to the constitutive relationship for the air phase (ie. equation [4]) gives,

$$[19] \qquad m_1^a\frac{\partial(\sigma-u_a)}{\partial t} + m_2^a\frac{\partial(u_a-u_w)}{\partial t} = \frac{-DR\theta}{w\,p}\frac{\partial^2 u_a}{\partial y^2} + \frac{(1-S)n}{p}\frac{\partial u_a}{\partial t}$$

The change of total stress with respect to time can be set to zero for the consolidation process. Simplifying and rearranging equation [19], the air phase partial differential equation can be written as follows.

$$[20] \qquad \frac{\partial u_a}{\partial t} = C_a \frac{\partial u_w}{\partial t} + c_v^a \frac{\partial^2 u_a}{\partial y^2}$$

where $C_a = \dfrac{m_2^a/m_1^a}{(1-m_2^a/m_1^a) + \dfrac{(1-S)n}{(u_a + u_{atm})\ m_1^a}}$ and is called the interactive constant associated with the air phase equation. This equation is further simplified by defining R_a as m_2^a/m_1^a.

$c_v^a = \dfrac{DR\theta}{w} \dfrac{1}{(1-R_a)\ (u_a+u_{atm})\ m_1^a + (1-S)\ n}$ and is called the coefficient of consolidation with respect to the air phase.

PORE PRESSURE BOUNDARY CONDITION EQUATIONS

When an external load is applied to an element of unsaturated soil, instantaneous compression occurs under undrained conditions and excess pressures are induced in the air and water phases. Two equations are necessary to predict the relative magnitudes of the excess pore air and pore water pressures. The pore pressures depend upon the compressibility of the soil structure, the air and water phases. In addition, the contractile skin has an effect on the relative changes in the pore air and pore water pressures. The induced pore pressures form the boundary conditions for the consolidation process.

The overall continuity requirement for the soil requires that the compression of the soil structure must equal the compression associated with the pore fluid phases. This is satisfied by equating equations [2] and [8] and using the water phase as the reference phase for the pore fluid pressure (7).

$$[21] \qquad m_1^s \, \Delta(\sigma - u_a) + m_2^s \, \Delta(u_a - u_w) = [S \, \beta_w + \frac{\Delta u_a}{\Delta u_w} \frac{(1-S)}{(u_a + u_{atm})}] \, n \, \Delta u_w$$

Defining $R_s = m_2^s / m_1^s$, equation [21] can be solved for the change in pore water pressure.

$$[22] \qquad \Delta u_w = [\frac{(R_s - 1) - \dfrac{(1-S)n}{(u_a + u_{atm}) \, m_1^s}}{R_s + S \, n \, \beta_w / m_1^s}] \, \Delta u_a + [\frac{1}{R_s + S \, n \, \beta_w / m_1^s}] \, \Delta\sigma$$

Equation [22] involves two unknowns (ie. Δu_w and Δu_a) and therefore another equation is required for its solution. A second equation is also logical since only one of two necessary constitutive relations has been used in formulating equation [22]. The second constitutive relation can be incorporated by considering the continuity of the air phase. Volume change described by the compression of the air phase must equal the volume change defined by the air phase constitutive relationship.

$$[23] \qquad m_1^a \, \Delta(\sigma - u_a) + m_2^a \, \Delta(u_a - u_w) = \frac{(1-S)n}{(u_a + u_{atm})} \, \Delta u_a$$

Simplifying equation [24] and solving for the change in pore air pressure gives,

$$[24] \qquad \Delta u_a = \left[\frac{R_a}{(R_a-1) - \frac{(1-S)\ n}{(u_a+u_{atm})\ m_1^a}}\right] \Delta u_w - \left[\frac{1}{(R_a-1) - \frac{(1-S)\ n}{(u_a+u_{atm})\ m_1^a}}\right] \Delta\sigma$$

Equations [22] and [24] can be solved for the change in pore air and pore water pressures resulting from a change in the applied load. Let us simplify these equations by defining the following variables.

$$[25] \qquad R_1 = \frac{(R_s-1) - \frac{(1-S)\ n}{(u_a+u_{atm})\ m_1^s}}{R_s + S\ n\ \beta_w/m_1^s}$$

$$[26] \qquad R_2 = \frac{1}{R_s + S\ n\ \beta_w/m_1^s}$$

$$[27] \qquad R_3 = \frac{R_a}{(R_a-1) - \frac{(1-S)\ n}{(u_a+u_{atm})\ m_1^a}}$$

$$[28] \qquad R_4 = \frac{1}{(R_a-1) - \frac{(1-S)\ n}{(u_a+u_{atm})\ m_1^a}}$$

Equations [22] and [24] can be written in terms of the above variables.

$$[29] \qquad \Delta u_w = R_1\ \Delta u_a + R_2\ \Delta\sigma$$

[30] $$\Delta u_a = R_3 \, \Delta u_w - R_4 \, \Delta\sigma$$

Equations [29] and [30] are combined and solved to give two pore pressure coefficients that can be used to compute the boundary conditions.

[31] $$B_w = \frac{\Delta u_w}{\Delta\sigma} = \frac{R_2 - R_1 \, R_4}{1 - R_1 \, R_3}$$

[32] $$B_a = \frac{\Delta u_a}{\Delta\sigma} = \frac{R_2 \, R_3 - R_4}{1 - R_1 R_3}$$

Equations [31] and [32] require an iterative technique in their solution since R_1, R_3 and R_4 contain the absolute pore air pressure value.

EXAMPLE PROBLEM

An example problem is solved to demonstrate the solution of boundary condition equations and the partial differential equations for the pore air and pore water phases. Initially, a compacted soil layer overlying an impervious surface is in equilibrium with the stresses shown in Figure 2. The initial equilibrium conditions are altered by changing the total stress applied to the top of the soil. The instantaneous change in total stress of 100 psi (689 kN/m^2) produced a corresponding pore air and pore water pressure change of 6 psi (41 kN/m^2) and 44 psi (303 kN/m^2), respectively. The initial and final stresses (and stress state variables) along with the assumed soil properties are shown in Table 1. The corresponding stress state variable changes on the constitutive surface are shown in Figure 3. The

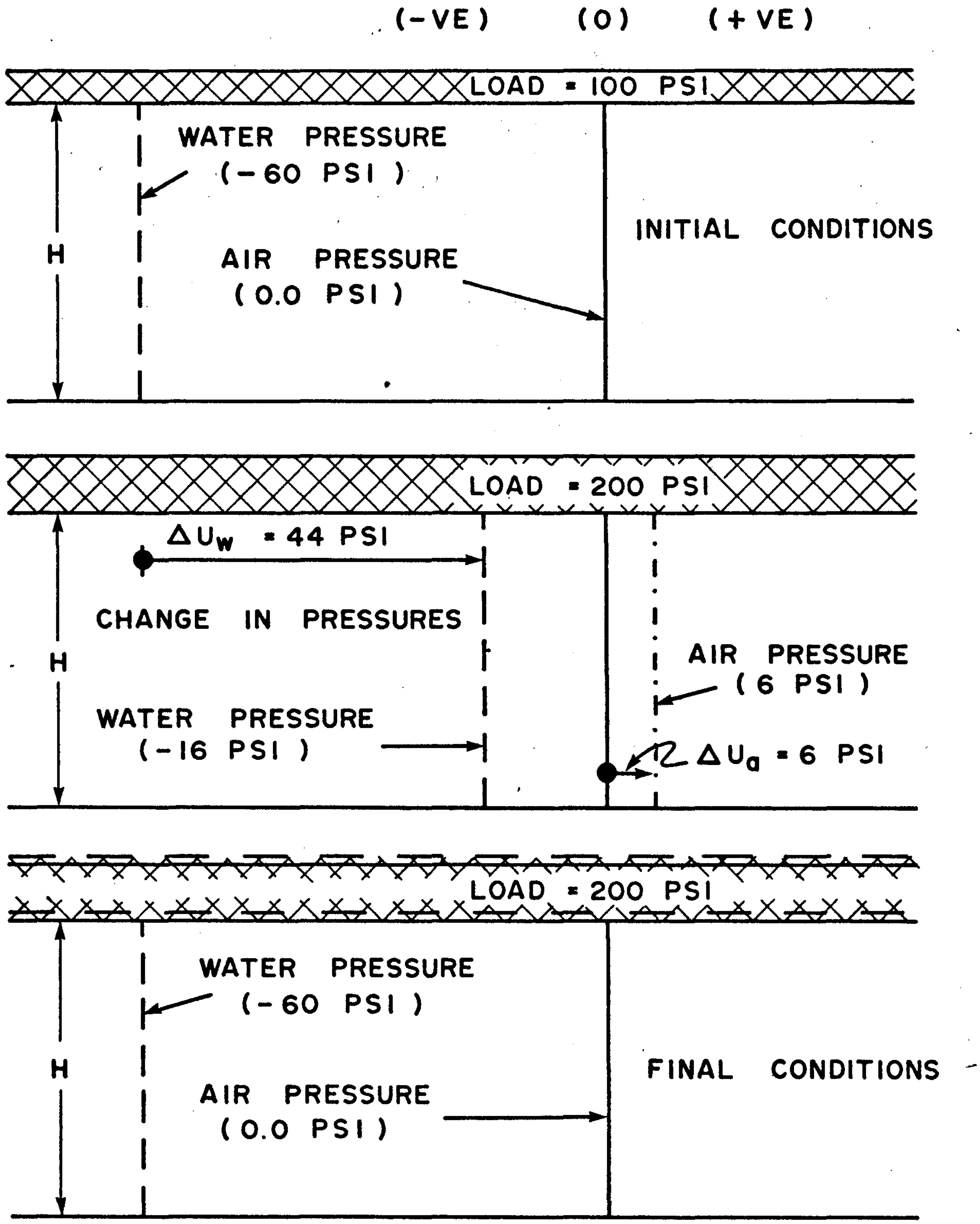

FIGURE 2 INITIAL AND FINAL BOUNDARY CONDITIONS FOR A CONSOLIDATION PROCESS

Table 1

Initial and Final Conditions of the Soil

Stresses	σ PSI	u_a PSI (gauge)	u_w PSI	$(\sigma - u_a)$ PSI	$(u_a - u_w)$ PSI
Initial Condition	100.0	0.0	-60.0	100.0	60.0
Instant After Loading Δu_a = +6 PSI, Δu_w = +44 PSI	200.0	6.0	-16.0	194.0	22.0
Final Conditions	200.0	0.0	-60.0	200.0	60.0

$S = 50\%$

$n = 50\%$

$m_1^a = 0.0088 \ in^2/lb$

$m_1^s = 0.001 \ in^2/lb$

$R_a = -0.01$

$R_w = 0.5$

$R_s = 0.5$

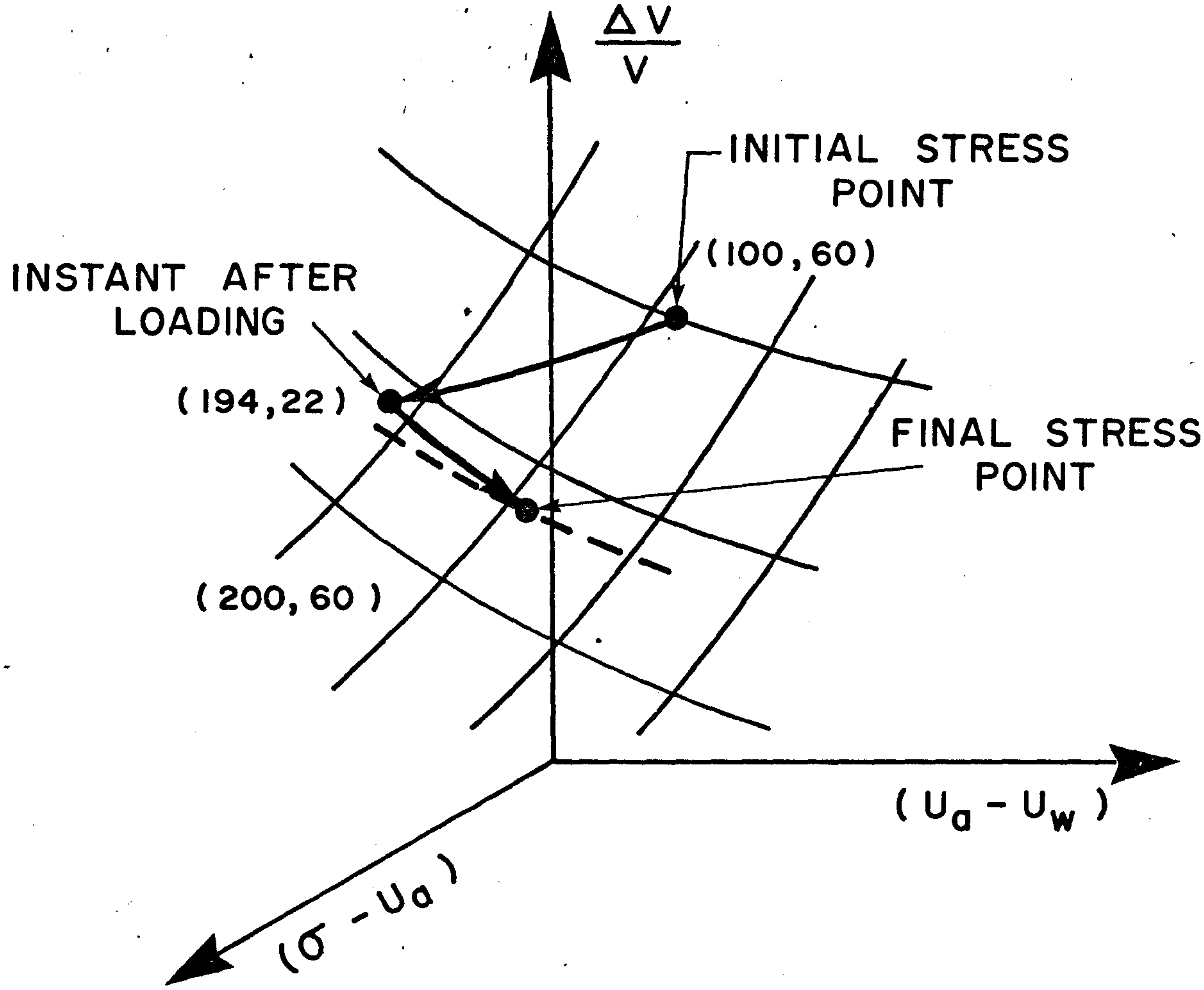

FIGURE 3 INITIAL AND FINAL STRESS POINTS ON A CONSTITUTIVE SURFACE FOR THE CONSOLIDATION PROCESS

dissipation of the excess pressure of the pore air and pore water phases is obtained by a simultaneous solution of equations [12] and [21] using the finite difference technique described in Appendix A.

The average degree of consolidation and the time factor are defined for each of the fluid phases. The average degree of consolidation with respect to the water phase is,

$$[33] \qquad U_w = 1 - \frac{\int_0^{2H} u_w \, dy}{\int_0^{2H} u_{wi} \, dy}$$

where U_w = average degree of consolidation with respect to the water phase

u_{wi} = initial water pressure

u_w = water pressure at any time

H = length of drainage path

The time factor with respect to the water phase is,

$$[34] \qquad T_w = c_v^w \, t/H^2$$

where t = elapsed time

Similarly, the average degree of consolidation and dimensionless time factor with respect to the air phase are defined as,

$$[35] \qquad U_a = 1 - \frac{\int_0^{2H} u_a \, dy}{\int_0^{2H} u_{ai} \, dy}$$

and

$$[36] \qquad T_a = c_v^a \, t/H^2$$

where U_a = average degree of consolidation with respect to the air phase

T_a = time factor with respect to the air phase

u_{ai} = initial air pressure

u_a = air pressure at any time

The average degree of consolidation with respect to the water phase (U_w) is plotted against the time factor for water phase (T_w) in Figure 4. It shows that the U_w versus T_w plot is slightly different than the conventional Terzaghi consolidation plot for saturated soils. This deviation of the U_w versus T_w plot from the conventional Terzaghi plot can be attributed to the interaction effect in the simultaneous solution of the two partial differential equations.

The average degree of consolidation with respect to the air phase (U_a) plotted against a time factor with respect to the air phase (T_a) is presented in Figure 5. It also shows a slight deviation from the Terzaghi solution for saturated soils. It should be noted that the c_v^a term has been assumed constant during the consolidation process. To do this, the absolute air pressure has been set to a constant equal to the average air pressure during the consolidation process. The error involved in this case is negligible.

Figure 6 shows the percent mid-plane pore pressures plotted against the dimensionless time factors for both of the fluid phases. Dimensionless

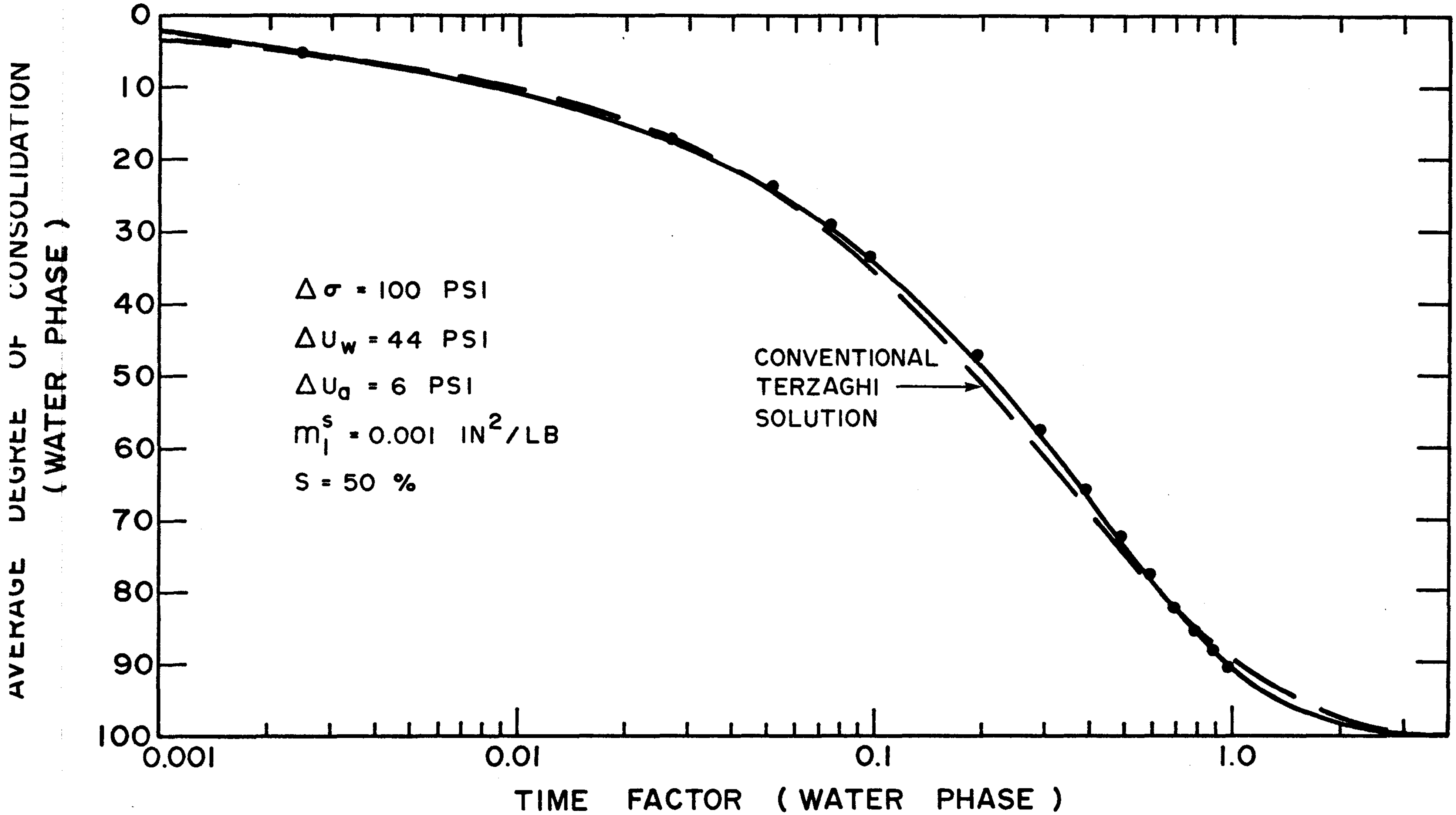

FIGURE 4 AVERAGE DEGREE OF CONSOLIDATION VERSUS TIME FACTOR

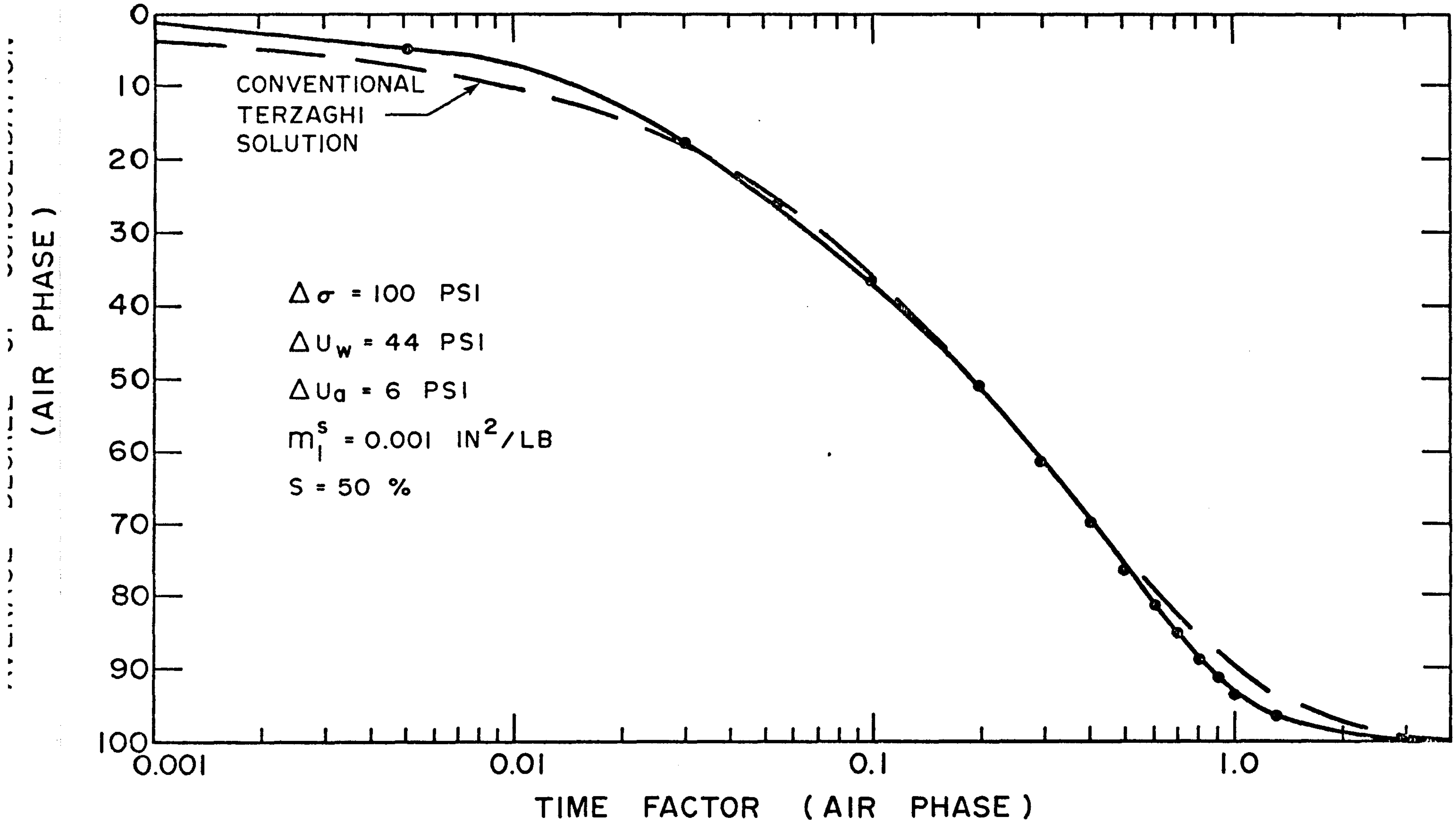

FIGURE 5 AVERAGE DEGREE OF CONSOLIDATION VERSUS TIME FACTOR

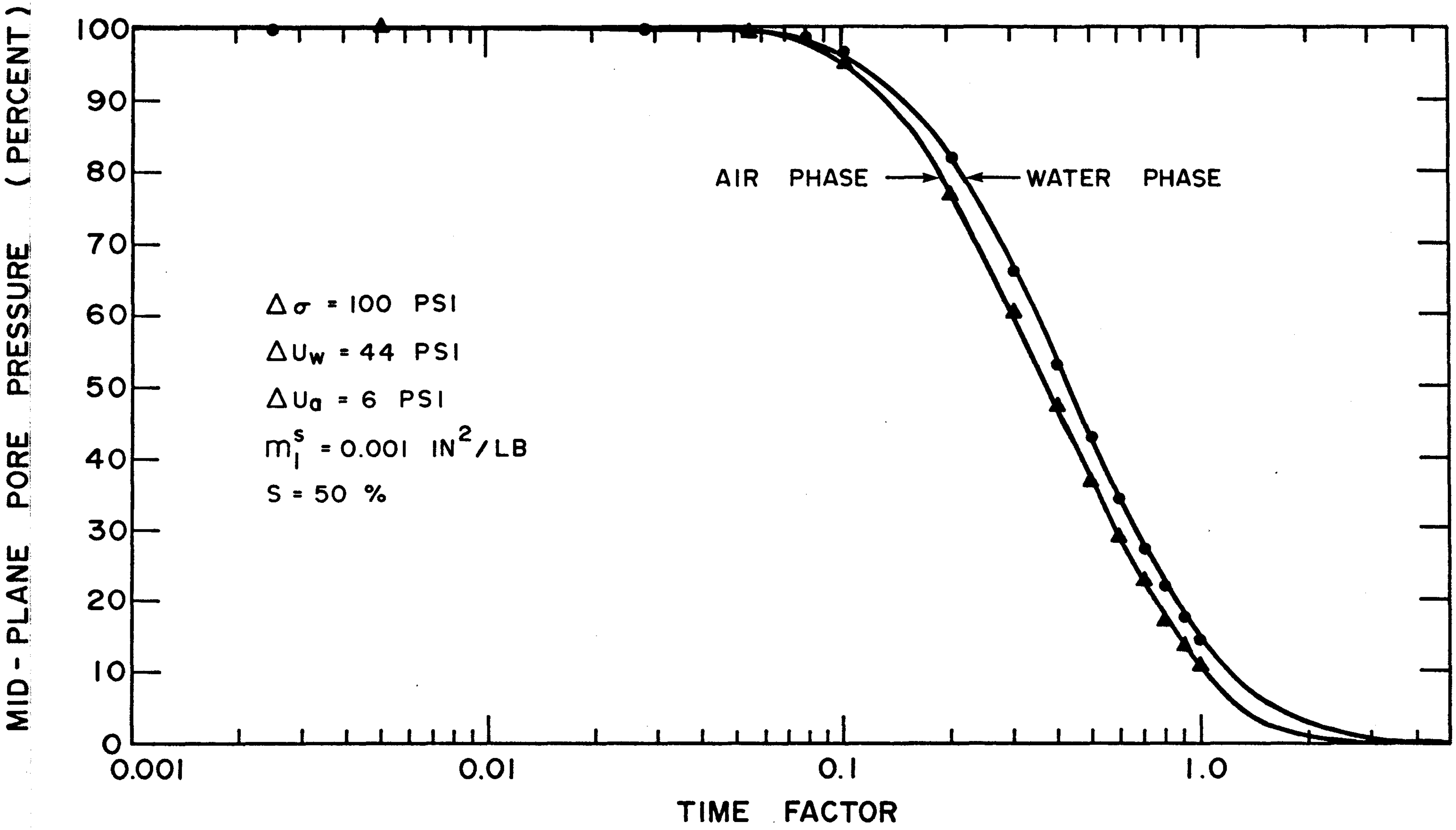

FIGURE 6 PERCENT MID - PLANE PORE AIR AND PORE WATER PRESSURE VERSUS TIME FACTOR

isochrones for both the water and the air phases are shown in Figure 7 and 8 respectively. The plots show only slight deviations from the conventional Terzaghi plots. Other example problems have shown the interaction effects between the air and water phase partial differential equations to be more extreme.

SUMMARY

All the necessary physical relationships are available for a complete formulation of the one-dimensional consolidation problem in unsaturated soils. The formulation proceeds along similar lines to that of Terzaghi's conventional theory for saturated soils. However, it is necessary to have one partial differential equation for the air phase and another partial differential equation for the water phase. They must be solved simultaneously.

$$\frac{\partial u_w}{\partial t} = -C_w \frac{\partial u_a}{\partial t} + c_v^w \frac{\partial^2 u_w}{\partial y^2}$$

$$\frac{\partial u_a}{\partial t} = C_a \frac{\partial u_w}{\partial t} + c_v^a \frac{\partial^2 u_a}{\partial y^2}$$

For the completely dry and saturated cases, there is a smooth transition to the conventional solutions. At intermediate degrees of saturation there is varying amounts of interaction between the air and water phases during the dissipation process.

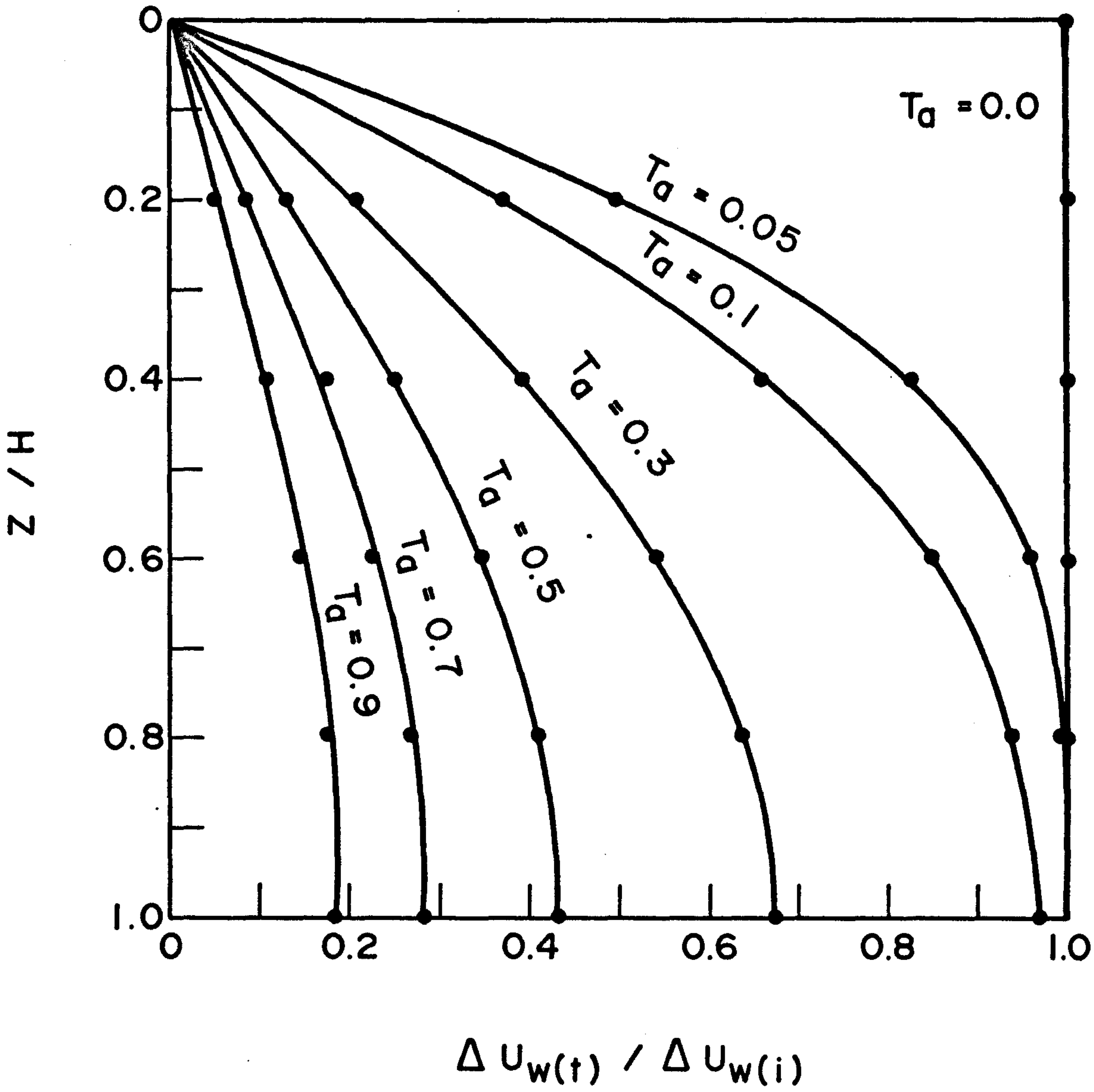

FIGURE 7 DIMENSIONLESS ISOCHRONES FOR THE WATER PHASE

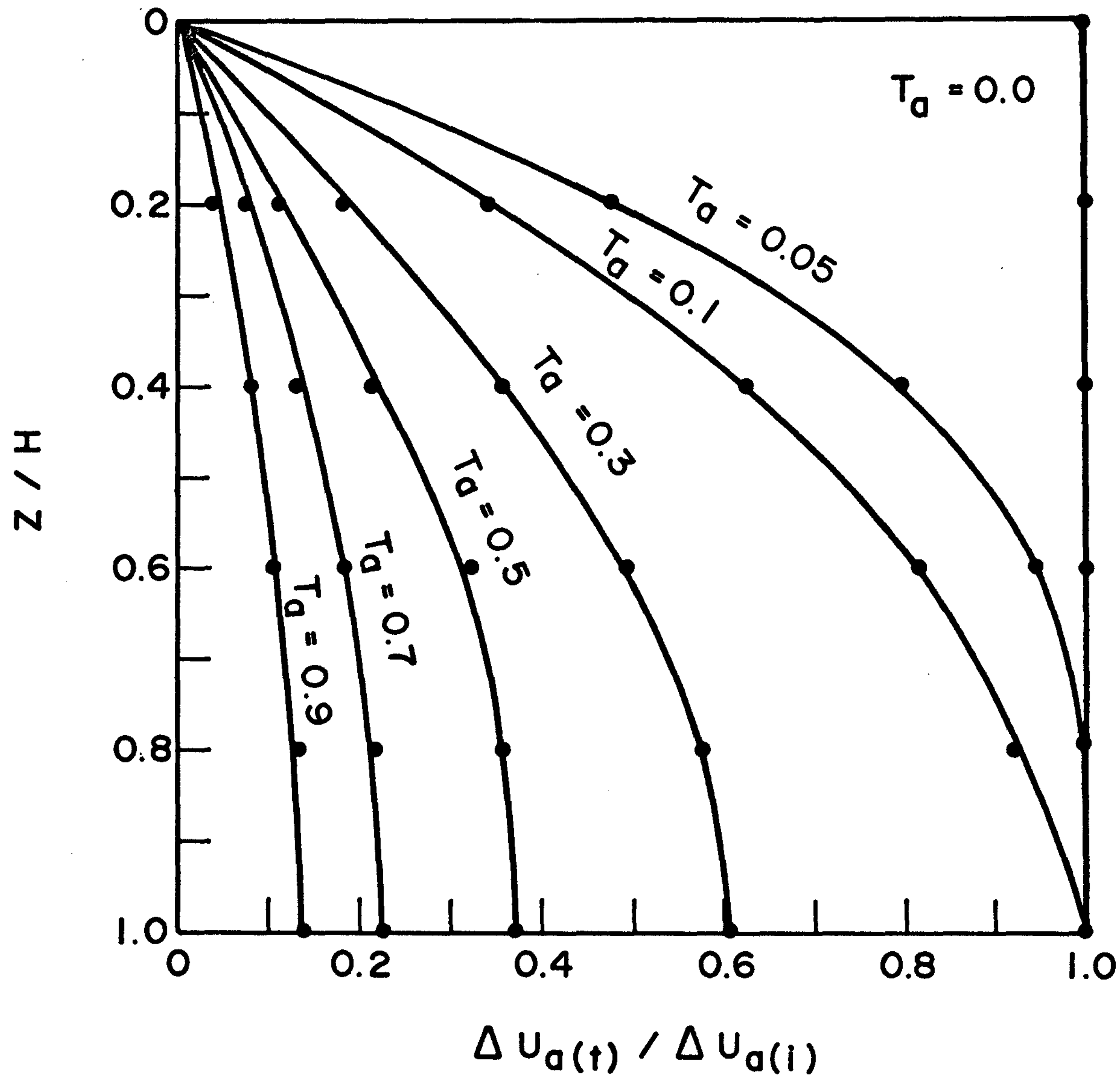

FIGURE 8 DIMENSIONLESS ISOCHRONES FOR THE AIR PHASE

The necessary physical relationships are also available for a rigorous formulation of the pore air and pore water pressures generated during undrained loading. Once again two equations are formulated that must be solved for the pore air and pore water boundary conditions to the consolidation problems.

$$\Delta u_w = \left[\frac{(R_s - 1) - \dfrac{(1-S)\, n}{(u_a + u_{atm})\, m_1^s}}{R_s + S\, n\, \beta_w / m_1^s} \right] \Delta u_a + \left[\frac{1}{R_s + S\, n\, \beta_w / m_1^s} \right] \Delta\sigma$$

$$\Delta u_a = \left[\frac{R_a}{(R_a - 1) - \dfrac{(1-S)\, n}{(u_a + u_{atm})\, m_1^s}} \right] \Delta u_w - \left[\frac{1}{(R_a - 1) - \dfrac{(1-S)\, n}{(u_a - u_{atm})\, m_1^a}} \right] \Delta\sigma$$

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Appendix A

An explicit finite difference technique is used to solve each of the partial differential equations. Since both equations must be solved simultaneously, the solution iterates between the two equations. In addition, one of the equations is non linear.

Figure A.1 shows the finite difference grids for the example problem and defines the related variables. Following are the finite difference forms for the partial differential equations.

Finite Difference Form of the Water Phase Partial Differential Equation

$$\text{[A1]} \qquad u_w(i,j+1) = u_w(i,j) - C_w\ (u_a(i,j+1) - u_a(i,j))$$

$$+ c_v^w \frac{\Delta t}{(\Delta z)^2} (u_w(i+1,j) + u_w(i-1,j) - 2u_w(i,j))$$

The $c_v^w\ \Delta t/(\Delta z)^2$ term has been set to 0.25 for the solution of the example problem.

Finite Difference Form of the Air Phase Partial Differential Equation

$$\text{[A2]} \qquad u_a(i,j+1) = u_a(i,j) + C_a\ (u_w(i,j+1) - u_w(i,j))$$

$$+ c_v^a \frac{\Delta t}{(\Delta z)^2} (u_a(i+1,j) + u_a(i-1,j) - 2u_a(i,j)$$

The $c_v^a\ \Delta t/(\Delta z)^2$ term has again been set to 0.25.

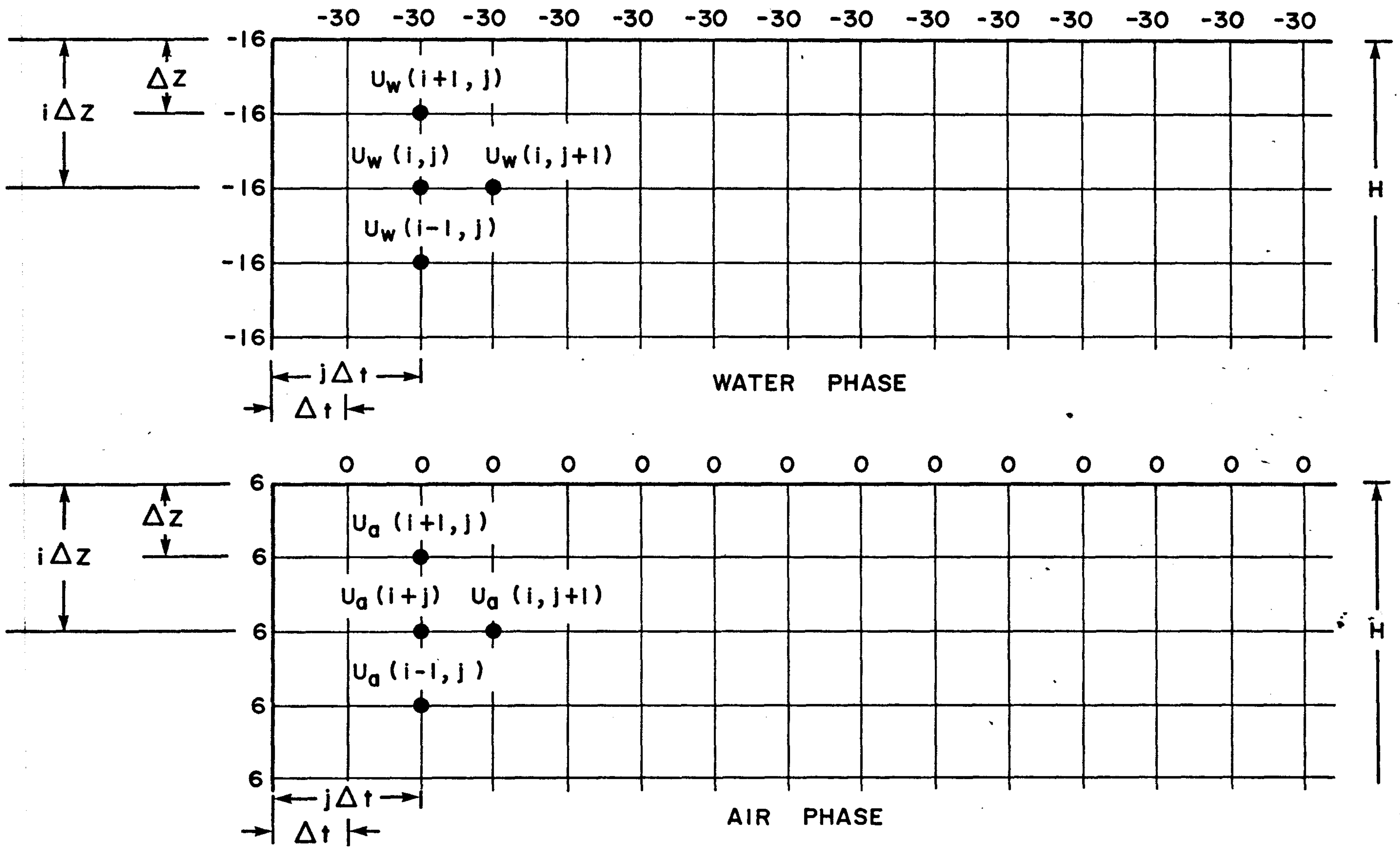

FIGURE A-1 FINITE DIFFERENCE MESH FOR THE CONSOLIDATION EQUATIONS

Procedure for Solving the Consolidation Equations

The procedure used in "marching" forward in time is as follows:

1.) The water phase equation is solved for the first time and depth step using an assumed air pressure value. Computations are commenced at the top of the soil layer where the boundary conditions are known.

2.) The air phase equation is solved for the first time and depth step using the water pressure calculated in step 1. It should be noted that the air phase equation has been linearized by assuming c_v^a to be constant.

3.) Steps 1.) and 2.) are repeated using the newly computed air pressure and water pressure, respectively.

4.) If the difference between the air pressure on subsequent iterations is greater than the designated tolerance, steps 1.) and 2.) are repeated. This continues until convergence is achieved.

5.) The above steps (ie. 1.) to 4.)) are then repeated for the next depth step.

6.) When computations for all the pressures associated with all depth steps is completed, the next time step proceeds according to the above steps.