Constitutive relations for volume change in unsaturated soils

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Volume change constitutive relations for unsaturated soils are proposed from a semi-empirical standpoint. One equation describes the deformation of the soil structure and a second equation defines the volume of water present in the element. Each equation can be viewed as a three-dimensional surface with two independent stress state variables forming the abscissas.

Uniqueness is tested by measuring volume changes resulting from stress changes in two orthogonal directions and comparing predicted and measured volume changes resulting from a stress change in a third direction. Samples of undisturbed Regina Clay and compacted kaolin showed good agreement between the predicted and measured volume changes for monotonic deformation of the soil structure. The agreement was not as close for the water phase. The variation was attributed to difficulties in measuring water volume changes over a long period of time. The laboratory results indicate that the proposed constitutive equations are of the appropriate form for use in engineering practice.

Des relations fondamentales de changement de volume pour les sols non saturés sont proposées d'un point de vue semi-empirique. Une équation décrit la déformation de la structure du sol et une seconde équation définit le volume d'eau présent dans l'élément. Chaque équation peut être envisagée comme une surface tridimensionnelle avec deux variables indépendantes d'état de contraintes constituant les abscisses.

L'unicité est contrôlée en mesurant les changements de volume résultant de changements de contraintes dans deux directions orthogonales et en comparant les changements de volume prédits et mesurés qui découlent d'un changement de contraintes dans une troisième direction. Des échantillons d'argile de Regina non remaniée et de kaolin compacté ont montré une bonne concordance entre les changements de volume prédits et mesurés pour une déformation monotone de la structure du sol. La concordance n'était pas si étroite pour la phase eau. La variation fut attribuée aux difficultés de mesure de changements de volume d'eau sur une longue période de temps. Les résultats de laboratoire indiquent que les équations fondamentales proposées sont de la forme appropriée à l'utilisation dans la pratique du génie.

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Introduction

Generally an unsaturated soil is considered to be a three-phase system. However, the independent properties and continuous bounding surfaces of the air–water boundary (*i.e.* contractile skin) require its consideration as a fourth phase (Davies and Rideal 1963). An element of unsaturated soil can therefore be considered as a mixture with two phases that come to equilibrium under applied stresses (*i.e.* soil particles and the contractile skin) and two phases that flow under applied pressures (*i.e.* the air and water). The air phase is assumed to be continuous.

Several effective stress equations have been proposed for unsaturated soils. However, they have lacked theoretical justification and have not proven satisfactory, as demonstrated by their limited application to engineering analyses. All equations have included a soil property in the description of the stress variable controlling soil behavior. Fredlund (1973*a*, *b*) used the principle of superimposition of coincident equilibrium stress fields to isolate the stress state variables associated with the soil particles and the contractile skin. The analysis indicated that any two of three possible normal stress variables can be used to define the stress state. Possible combinations are:

(i) $(\sigma - u_a)$ and $(u_a - u_w)$	(ii) $(\sigma - u_w)$ and $(u_a - u_w)$	(iii) $(\sigma - u_w)$ and $(\sigma - u_a)$

where σ = total normal stress, u_a = air pressure, and u_w = water pressure.

Null experiments (*i.e.* $\Delta\sigma = \Delta u_a = \Delta u_w$) supported the theoretically proposed stress state variables (Fredlund 1973*a*).

The deformation variables required to describe the changes in each phase in an element can be derived from the continuity requirement for a multi-phase continuum (Fredlund 1973*a*, 1974). If the soil particles are assumed incompressible and the volume change of the contractile skin assumed internal to the element, the continuity requirement for an element reduces to,

$$[1] \qquad \frac{\Delta V}{V} = \frac{\Delta V_w}{V} + \frac{\Delta V_a}{V}$$

where V = total volume, V_w = volume of water in the element, and V_a = volume of air in the element.

The measurement or prediction of any two of the above volume changes allows complete monitoring of the volume–weight change relationships. The amounts of air and water in the element can be represented respectively by one volumetric deformation state variable:

$$[2] \qquad \theta_a = \frac{\Delta V_a}{V}$$

$$[3] \qquad \theta_w = \frac{\Delta V_w}{V}$$

The change in volume of the overall element (*i.e.* soil structure) can be written as the sum of the normal strain components.

$$[4] \qquad \epsilon = \frac{\Delta V}{V} = \epsilon_x + \epsilon_y + \epsilon_z$$

where ϵ = volumetric strain; ϵ_x, ϵ_y, ϵ_z = normal strain components in the x, y, and z directions, respectively.

The independently derived stress and deformation state variables are linked by suitable constitutive relations. In the case of unsaturated soils, these are generally proposed from a semi-empirical standpoint and must be checked experimentally for uniqueness.

Literature Review

In Biot's (1941) considerations of three-dimensional consolidation, the soil structure was assumed to behave as a linear, reversible, isotropic material. The soil was also assumed to be unsaturated in that the pore water contained occluded air bubbles. The deformation was described in terms of the effective stress state variable $(\sigma - u_w)$ and the pore fluid pressure, u_w. Biot (1941) also proposed a second constitutive relationship to describe the dependence of water content changes on the above stress variables. In total, four distinct physical soil properties were required to link the stress and deformation state variables.

Attempts to link the deformation of an unsaturated soil with Bishop's effective stress equation have met with limited success (Jennings and Burland 1962). Bishop and Blight (1963) recognized these problems and concluded that the stress path of both components (*i.e.* $(\sigma - u_a)$ and $(u_a - u_w)$) must be considered. They suggested plotting volume change data *vs.* both stress variables to form a three-dimensional plot. Burland (1965) reiterated his dissatisfaction with Bishop's effective stress equation and proposed a set of constitutive relationships for the soil structure and the water phase.

Aitchison (1967) realized the importance of mapping volume changes with respect to the independent stress variables. Aitchison (1969) presented typical volume change curves obtained by independently following the $(\sigma - u_a)$ and $(u_a - u_w)$ stress paths.

Matyas and Radhakrishna (1968) performed tests on identically prepared mixtures of 80% flint powder and 20% kaolin in which they controlled the total, air and water pressures in isotropic and K_o compression tests. The constitutive surfaces of void ratio and degree of saturation *vs.* the $(\sigma - u_a)$ and $(u_a - u_w)$ stresses were traced out using different stress paths to test their uniqueness. When the $(u_a - u_w)$ stress was *decreased* or the $(\sigma - u_a)$ stress was *increased*, the void ratio results traced out a single warped surface with the soil structure always decreasing in volume. Normally, a reduction in suction $(u_a - u_w)$, causes the soil to swell, but their results show overall volume decrease, indicating a metastable structured soil. Even though the soil structure indicates a collapse phenomenon, their results show a unique (*i.e.* soil structure) constitutive surface. The degree of saturation was always an increasing variable. When other stress paths were considered, the void ratio *vs.* stress constitutive surface was not completely unique. Matyas and Radhakrishna (1968) state that the "restrictions on the paths arise from the fact that the hysteresis on the soil structure due to loading and unloading, wetting and drying, introduce certain non-unique characteristics." Their degree of saturation surfaces were not unique and they attributed the deviations to "incomplete saturation during the wetting process."

Barden *et al.* (1969) performed tests on low to high plasticity illite clay samples. The total air and water pressures were controlled while investigating the effect of various stress paths during K_o loading conditions. In all cases the applied stress $(\sigma - u_a)$ was *increased* subsequent to the initial conditions. In most cases the suction $(u_a - u_w)$ was *increased* subsequent to the initial conditions, however, in a few cases it was decreased. Their results indicate that the soil structure volume change is dependent upon the direction of saturation. Barden *et al.* (1969) conclude "that hysteresis between saturation and desaturation processes is the major cause of stress path dependence." They suggest that there is also a strong stress path dependence of the degree of saturation surface upon the reversal of the direction of saturation.

Brackley (1971) used two independent stress variables (*i.e.* $(\sigma - u_a)$ and $(u_a - u_w)$) in his study of partial collapse of unsaturated expansive clays. Difficulties were encountered in attempts to apply Bishop's effective stress equation. Subsequently, the description of volume change behavior was proposed as a function of the independent stress variables.

In this paper, the constitutive relations for an unsaturated soil are examined in the following manner:

(i) The form and number of constitutive relationships for an unsaturated soil are proposed from a semi-empirical standpoint.

(ii) The constitutive surface is experimentally tested for uniqueness near a point. In other words, is the constitutive surface unique for small stress changes in various directions from a stress point? Limitations to uniqueness are established.

Proposed Constitutive Relations

(a) Soil Structure

Insight into the constitutive relations for the soil structure may be obtained by inspecting incremental relations for a linear, elastic, isotropic material. Selecting $(\sigma - u_w)$ and $(u_a - u_w)$ as the stress state variables, the normal strain in the x direction, ϵ_x is,

$$[5] \quad \epsilon_x = \frac{(\sigma_x - u_w)}{E_1} - \frac{\mu_1}{E_1}\cdot(\sigma_y + \sigma_z - 2\cdot u_w) + \frac{(u_a - u_w)}{H_1}$$

where E_1 = an elastic modulus with respect to a change in $(\sigma - u_w)$, μ_1 = Poisson's ratio with respect to relative strains in the x and y (or z) directions, and H_1 = an elastic modulus with respect to a change in $(u_a - u_w)$.

Similar equations can be written for the y and z directions. The equations are essentially the same as those proposed by Biot (1941) and Coleman (1962). The volumetric strain of the soil structure, ϵ, is equal to the sum of the normal strain components [4].

$$[6] \quad \epsilon = C_t\cdot d(\sigma - u_w) + C_a\cdot d(u_a - u_w)$$

where C_t = compressibility of the soil structure with respect to a change in $(\sigma - u_w)$. For isotropic loading, σ is the all round stress and;

$$C_t = 3\cdot\left[\frac{1 - 2\cdot\mu_1}{E_1}\right]$$

For K_o loading, σ is the major principal stress and;

$$C_t = \frac{(1 + \mu_1)(1 - 2\cdot\mu_1)}{E_1\cdot(1 - \mu_1)}$$

C_a = compressibility of the soil structure with respect to a change in $(u_a - u_w)$. For isotropic loading:

$$C_a = 3/H_1$$

For K_o loading;

$$C_a = \frac{(1 - \mu_1)}{H_1\cdot(1 + \mu_1)}$$

The above incremental equation [6] can be written in a more general form to account for non-linear properties.

$$[7] \quad \epsilon = \frac{1}{v}\cdot\frac{\partial v}{\partial(\sigma - u_w)}\cdot d(\sigma - u_w) + \frac{1}{v}\cdot\frac{\partial v}{\partial(u_a - u_w)}\cdot d(u_a - u_w)$$

where v = unit volume

$\frac{1}{v}\cdot\frac{\partial v}{\partial(\sigma - u_w)}$ = compressibility of the soil structure when $d(u_a - u_w)$ is zero

$\frac{1}{v}\cdot\frac{\partial v}{\partial(u_a - u_w)}$ = compressibility of the soil structure when $d(\sigma - u_w)$ is zero

All the above equations can be written using other combinations of stress state variables (Fig. 1a). Using $(\sigma - u_a)$ and $(u_a - u_w)$, the general equation for the soil structure takes the form,

$$[8] \quad \epsilon = \frac{1}{v}\cdot\frac{\partial v}{\partial(\sigma - u_a)}\cdot d(\sigma - u_a) + \frac{1}{v}\cdot\frac{\partial v}{\partial(u_a - u_w)}\cdot d(u_a - u_w)$$

The stress state variables $(\sigma - u_a)$ and $(\sigma - u_w)$ can also be used to form a similar equation. Any one of the above three forms can be used; however, only one of the equations is independent.

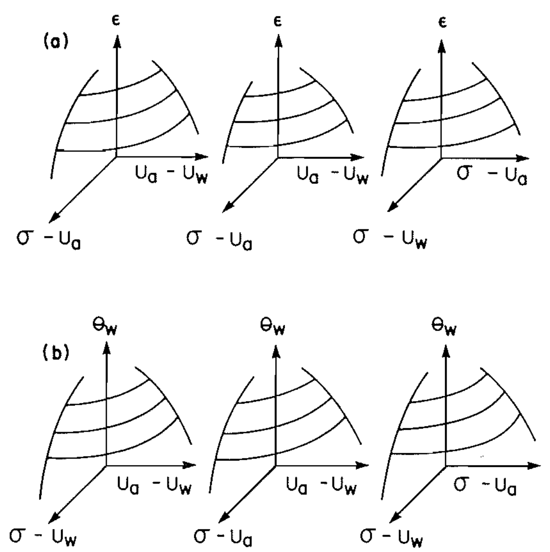

FIG. 1. Graphical representation of the constitutive surfaces for an unsaturated soil. (a) Constitutive surface for the soil structure. (b) Constitutive surface for the water phase.

It must be emphasized that the soil structure (or void ratio) constitutive relationship is not sufficient to completely describe the change in state of an unsaturated soil. Either the air or water phase constitutive relations must also be formulated. Due to the compressible nature of the air phase, it is recommended that the water phase (or water content) constitutive relationship be formulated. The water content *vs.* soil suction equation generally used in the soil physics area (Childs 1969) becomes a special case of the total description of state.

(b) Water Phase

The water phase constitutive relation describes the volume of water present in the referential soil structure element. The water is assumed to be incompressible and the equation accounts for the net inflow or outflow of water from the element. The water phase constitutive relation can be formulated semi-empirically on the basis of a linear combination of the state variables.

$$[9] \quad \theta_w = \frac{(\sigma_x + \sigma_y + \sigma_z - 3\cdot u_w)}{3H_1} + \frac{(u_a - u_w)}{R_1}$$

where θ_w = net inflow or outflow of water, R_1 = a water volumetric modulus associated with a change in $(u_a - u_w)$, and H_1 = a water volumetric modulus associated with a change in $(\sigma - u_w)$.

Equation [9] can be written in a more general form to account for nonlinear soil properties.

$$[10] \quad \theta_w = \frac{1}{v}\cdot\frac{\partial v_w}{\partial(\sigma - u_w)}\cdot d(\sigma - u_w) + \frac{1}{v}\cdot\frac{\partial v_w}{\partial(u_a - u_w)}\cdot d(u_a - u_w)$$

where v_w = volume of water in the element

$\frac{1}{v}\cdot\frac{\partial v_w}{\partial(\sigma - u_w)}$ = slope of the water volume *vs.* $(\sigma - u_w)$ plot when $d(u_a - u_w)$ is zero

$\frac{1}{v}\cdot\frac{\partial v_w}{\partial(u_a - u_w)}$ = slope of the water volume *vs.* $(u_a - u_w)$ plot when $d(\sigma - u_w)$ is zero.

Equation [10] applies for isotropic and K_o loading; however, the slopes contain different combinations of the water volumetric moduli. Other combinations of stress state variables can also be used to derive the volumetric representation of the change in the amount of water in the element (Fig. 1b). The specialization of [10] used in soil physics relates water content to changes in matrix suction ($u_a - u_w$).

(c) Air Phase

The change in volume of air present in an element can be written as the difference between the soil structure volume change and the change in the volume of water present in the element.

Sign Convention for Deformation Soil Properties

A positive or negative sign must be associated with each deformation (or compressibility) modulus since the direction of deformation is not necessarily known. That is, an increase in a stress state variable does not always produce a volume change in the same direction. For example, a decrease in suction ($u_a - u_w$) in a stable structured soil will result in swelling, whereas, it may cause a volume decrease in a meta-stable soil (Barden *et al.* 1969). Even for a particular soil, the direction of deformation is not fixed with respect to a single stress state variable. Rather, it can depend upon the component of the stress state variable that is changed. Consider, for example, the ($u_a - u_w$) stress variable and the deformation of the soil structure. The suction can be increased by either increasing the air pressure or decreasing the water pressure. If the air pressure is increased in an unsaturated soil, the soil structure often expands. If the water pressure is decreased, the soil structure generally compresses. Therefore, when establishing the direction of deformation, it is necessary to either establish the stress component being changed (*i.e.* u_a or u_w) or else simultaneously consider the stress state variables affected (*e.g.* ($\sigma - u_w$) and ($u - u_w$)).

The suggested sign convention is as follows: if an *increase* in the stress state variable produces a phase volume *decrease*, the deformation modulus is *POSITIVE*. If the stress variable *increase* produces a volume *increase*, the compressibility modulus is *NEGATIVE*. The same sign convention is applicable to the soil structure, air and water phases.

Procedure to Experimentally Verify the Uniqueness of the Constitutive Surface

Two constitutive surfaces are required for the complete description of volume changes sustained by an unsaturated soil. Their uniqueness can be explored; first, in terms of small deviations from a stress point and second, in terms of larger stress state variable changes and reversals of stress. This paper examines only the uniqueness of the constitutive relations near a stress point. The term 'uniqueness' is used to indicate that there is one and only one relationship between state variables. However, if complete 'uniqueness' does not exist, the term 'uniqueness' can be used in a more restrictive sense. For large stress changes, the drying and wetting of soils introduces hysteresis into the constitutive surfaces. In the restricted sense, the term 'uniqueness' can be applied to either the drying surface or the wetting surface. The water phase constitutive surface can be tested for 'uniqueness' in a similar manner.

Let us suppose that it is possible to prepare several 'identical' unsaturated soil samples that are subjected to the same total pressure and the same air and water pressures. Therefore, all samples are at the same stress point in space and have the same initial volume–weight properties. Then let each sample be subjected to varying stress changes, while monitoring the volume changes associated with each phase (Fig. 2). If the constitutive surface is essentially planar near a stress point, the deformation moduli associated with any orthogonal directions can be used to describe the deforma-

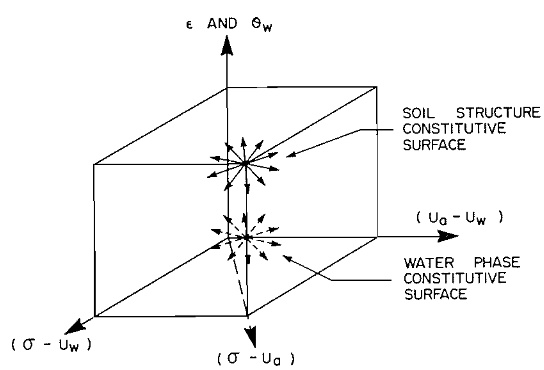

FIG. 2. Ideal tests to prove uniqueness of the soil structure and water phase constitutive surface at a point.

tions produced by other changes in the stress variables.

The above tests are somewhat fictitious since they would be difficult and extremely time consuming to conduct. Therefore, a simpler and more rapid procedure was adopted. One sample was subjected to small stress increments along three stress paths. Using the deformations from any two of the increments, it is possible to compute two corresponding compressibility moduli. Now the deformation equation can be used to compute the anticipated deformation along any other stress path. The computed deformation is compared with the measured deformation. The constitutive surface at a point is unique if the measured and predicted deformations are essentially equal in all cases.

Three suitable stress paths are traced by changing the total, air and water pressures and allowing equalization after each pressure change (Fig. 3). The uniqueness tests can be applied to any of the proposed constitutive equations (*i.e.* soil structure, air and water phases). As an example, let us consider the soil structure deformations for the above case. One form of the constitutive equation for the soil structure is,

$$\frac{\mathrm{d}v}{v} = C_{\mathrm{t}} \cdot \mathrm{d}(\sigma - u_{\mathrm{w}}) + C_{\mathrm{a}} \cdot \mathrm{d}(u_{\mathrm{a}} - u_{\mathrm{w}})$$

Suppose the first pressure increment is a decrease in the total stress. Therefore,

$$C_{\mathrm{t}} = \frac{\mathrm{d}v/v}{\mathrm{d}(\sigma - u_{\mathrm{w}})}$$

Let the second pressure change be an increase in the air presure. Therefore,

$$C_{\mathrm{a}} = \frac{\mathrm{d}v/v}{\mathrm{d}(u_{\mathrm{a}} - u_{\mathrm{w}})}$$

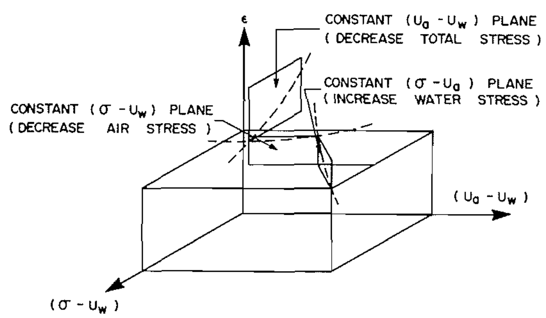

FIG. 3. Stress path adhered to for the uniqueness tests at a point.

Now that the two compressibility moduli are known (*i.e.* C_{t} and C_{a}), the volume change corresponding to a change in the pore water pressure can be computed. This value can be compared with the measured volume change. Agreement indicates uniqueness of the constitutive surface at a point. Some discrepancy is anticipated since all stress changes do not initiate from the same stress point.

The procedure used in most of the tests was as follows. The sample was allowed to come to equilibrium under an arbitrary set of stress conditions. An attempt was made to have a large initial suction. The water pressure was generally increased for the first pressure increment. After equalization, the air pressure was decreased. After equalization, the total pressure was decreased. The above procedure was repeated through several cycles as long as the suction remained positive. Throughout these steps, the water phase volume was continually increasing. The soil structure and air phase generally underwent a reversal in their direction of volume change. The soil structure and water phase constitutive surfaces are analyzed in detail since they were measured independently in the laboratory.

Description of Equipment

Four pieces of equipment were used in the examination of volume change behavior of unsaturated soils (Fredlund 1973*a*). Two employed modified Anteus oedometers[1] for one-dimensional strain conditions. The specimens were $2\frac{1}{2}$ in. (6.4 cm) in diameter and approximately 1 in. (2.5 cm) in height. The other two pieces of equipment allowed isotropic volume change testing conditions in modified Wykeham Farrance triaxial cells. The specimens were 4 in. (10.2 cm) in diameter and approximately $2\frac{1}{2}$ in. (6.4 cm) in height.

The air and water pressures were separated by means of a high air entry disc placed at the bottom of the sample. Distilled, de-aired water was used below the high air entry disc. Al-

[1]The Anteus oedometers are manufactured by Testlab Corporation in the United States.

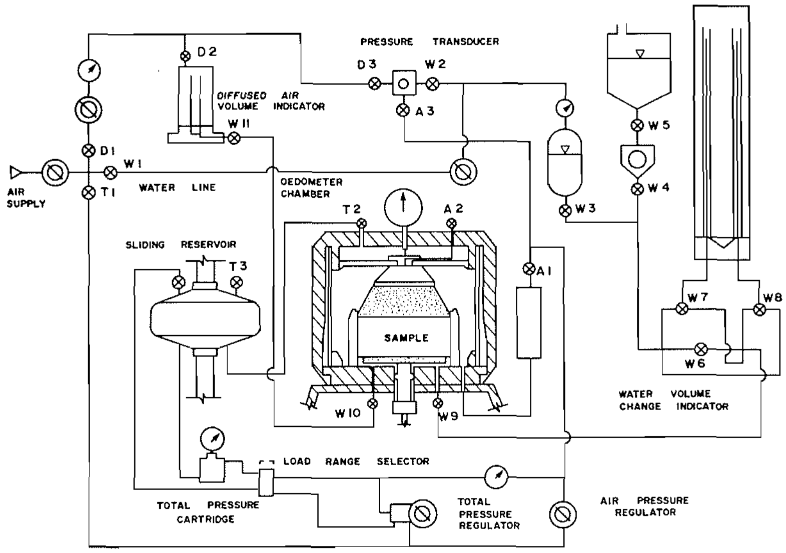

FIG. 4. Layout of modified Anteus oedometer.

though the high air entry discs do not leak air at pressures less than their air entry value, dissolved air diffuses through the water in the disc and collects below the base of the disc. For accurate water volume change measurements, it was necessary to measure the volume of diffused air and apply the appropriate correction. The diffused volume was measured by periodically flushing the base plate and measuring the volume of air displacing water in an inverted burette (Fredlund 1975).

The Anteus oedometer was selected for modification due to its versatile design. Figure 4 shows the layout of the modified Anteus oedometer and associated plumbing equipment. The chamber of the oedometer (normally filled with water for back pressuring the water phase of soils) was filled with air to regulate the air pressure in the soil sample. All pressures were controlled using pressure regulators with an accuracy better than ±0.02 psi (0.14 kN/m^2). The valves on Figure 4 associated with the total air, water, and diffused air pressure control are labelled with a T, A, W, and D, respectively.

The modified Wykeham Farrance triaxial cell is shown in Fig. 5. Linear voltage displacement transducers (LVDT's) were used to measure the vertical and lateral displacements. The cell was pressurized with air. A composite membrane consisting of slotted aluminum foil, vacuum grease, and two latex membranes was used to prevent the diffusion of air from the cell into the sample.

Presentation of Test Data

Four series of experiments were run to test the constitutive surfaces for uniqueness. The first three samples (No. 15A, 16, and 17) were undisturbed Regina Clay, while sample No. 21 was compacted kaolin. Samples No. 15A and 17 were from a depth of 2 ft (0.6 m) and sample No. 16 was from a depth of 15 ft (4.6 m) below a grassy area in northwest Regina, Saskatchewan. The shallow samples exhibited a very nuggetty macro-fissured structure due to desiccation (Fredlund 1973*a*). The properties of the soils tested are shown in Table 1.

Table 2 summarizes the initial volume–weight properties and stress conditions at the start of each pressure change.

The volume changes observed as a result of the stress component changes are summarized in Table 3. The volume changes at two elapsed times are considered for each sample since it is difficult to determine the point of complete equalization.

The object of the uniqueness analysis is to compute two compressibility values and use them to predict volume change in a third direction. The predicted volume change can be compared with a laboratory measured value.

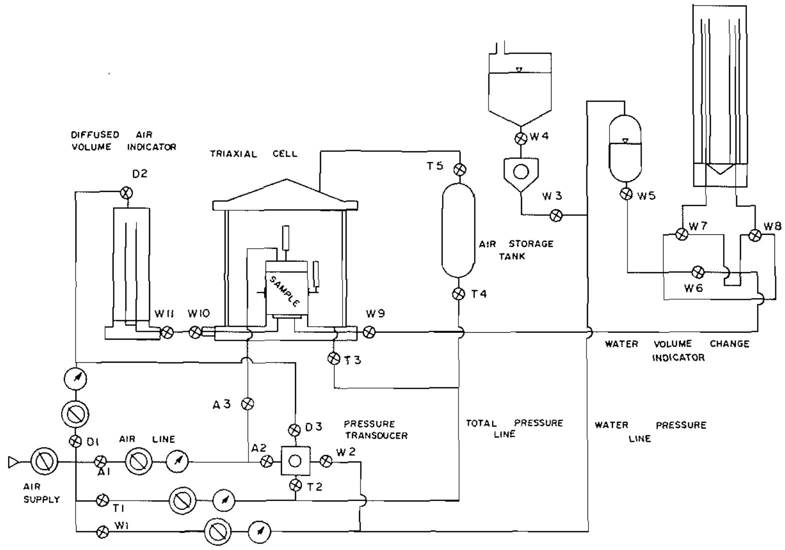

FIG. 5. Layout of modified triaxial apparatus.

TABLE 1. Soil properties of samples used for uniqueness tests

Soil type	Sample numbers	Specific gravity	Liquid limit (%)	Plastic limit (%)	Sand sizes (%)	Silt sizes (%)	Clay sizes (%)
	15A	2.78	75.0	26.0	5	30	65
Regina clay	16	2.78	77.0	26.8	0	45	55
	17	2.78	76.3	25.4	—	—	—
Kaolin	21	2.62	64.2	34.7	2	35	63

Comparison of Predicted and Measured Volume Changes

The predicted and measured volume changes can be visually assessed by plotting the predicted volume changes *vs.* the measured volume changes. In addition, a statistical correlation between the predicted and measured volume changes helps quantify the agreement. The degree of correlation can be expressed in terms of the correlation coefficient, the slope, and intercept of the best-fit straight line (Neville and Kennedy 1966). Uniqueness would imply a slope of 1.00 and an intercept of 0.00. The correlation coefficient is a measure of the dispersion about the best-fit line. The significance of the correlation coefficient can be evaluated by comparing the computed value with the value that can be expected at a given level of significance. A high correlation coefficient implies causation and the existence of a reliable mathematical model for the constitutive surface.

Soil Structure

Figure 6 shows the agreement between the measured volume changes and the predicted volume changes for sample No. 15A. Sample No. 15A was tested under K_0 loading conditions. For elapsed times of 1000 and 5000 min, all points fall essentially along the 45° line, indicating virtually perfect correlation.

Sample No. 16 (Fig. 7), tested under isotropic loading conditions, shows considerably more dispersion in the data. The correlation on the results at an elapsed time of 15 000 min are superior to those at 2000 min. For some stress changes, it is possible to compare the predicted volume change with both a previous and subsequent measured volume change. In these cases, the graph shows a bar between the

TABLE 2. Initial volume–weight properties and initial stress conditions for uniqueness tests

Test No.	Initial pressures σ (psi)	u_a (psi)	u_w (psi)	Water content (%)	Void ratio	Degree of saturation (%)
Sample No. 15A						
C-1	77.71	50.01	1.47	18.46	0.7990	64.21
C-2	77.91	50.14	12.19	21.56	0.8020	74.73
C-3	77.92	40.15	12.11	21.70	0.8014	75.28
C-4	67.63	40.16	12.11	22.09	0.8058	76.20
C-5	67.89	40.16	21.92	23.04	0.8106	79.02
C-6	67.92	30.19	22.21	23.54	0.8099	80.80
Sample No. 16						
C-7	71.46	50.18	3.66	26.28	0.8354	87.44
C-8	71.39	50.16	13.57	26.51	0.8374	88.01
C-9	71.30	40.04	13.48	26.69	0.8321	89.18
C-10	61.11	39.72	13.18	26.92	0.8364	89.47
C-11	61.29	39.93	23.25	27.23	0.8404	90.08
C-12	61.44	30.11	23.39	27.68	0.8355	92.09
C-13	51.65	30.29	23.52	28.08	0.8435	92.53
Sample No. 17						
C-14	77.72	50.01	1.50	22.51	0.7510	83.31
C-15	77.76	49.98	11.50	23.03	0.7577	84.48
C-16	77.84	39.99	11.45	23.20	0.7582	85.07
C-17	67.67	39.99	11.37	23.66	0.7646	86.04
Sample No. 21						
C-18	82.04	61.20	4.36	31.60	1.0425	79.92
C-19	71.97	60.93	4.08	31.61	1.0441	79.86
C-20	72.06	51.09	4.22	31.56	1.0445	79.84
C-21	72.02	51.00	14.20	31.71	1.0453	79.81
C-22	62.41	51.23	14.24	31.83	1.0508	79.60
C-23	62.64	41.58	14.43	31.91	1.0510	79.59
C-24	62.46	41.34	24.13	32.29	1.0543	79.46
C-25	52.53	41.16	23.80	32.56	1.0717	78.79
C-26	52.53	31.13	23.80	33.26	1.0681	78.93
C-27	52.25	30.86	28.40	34.45	1.0682	78.94
C-28	43.06	31.46	28.91	35.10	1.0830	78.37
C-29	52.06	31.13	28.63	35.04	1.0749	78.67
C-30	52.02	31.09	23.76	34.32	1.0696	78.87
C-31	81.35	31.27	23.85	33.17	1.0251	80.61
C-32	81.07	61.07	23.66	32.44	1.0260	80.57

two comparisons. The accuracy of the volume measuring devices was such that it should not have caused discrepancies in the uniqueness analysis. The scatter in the results must be considered in terms of the demanding requirements of the test for uniqueness. For example, it would be anticipated that the effects of nonlinearity of the constitutive surface and hysteresis are factors that could introduce significant scatter. These will be dealt with in more detail later.

Sample No. 17 (Fig. 8) was tested under K_0 loading conditions. There is a good correlation between the predicted and measured soil structure volume changes. However, there are only a limited number of comparisons that can be made.

Sample No. 21 was a compacted kaolin sample tested under isotropic loading. The results indicated a low correlation. They are reviewed later in terms of the effect of nonlinearity and hysteresis.

Table 4 shows a summary of the statistical properties for the soil structure, assuming a linear constitutive surface locally. Samples No. 15A, 16, and 17 show correlation coefficients exceeding the critical correlation coefficient for a one percent level of significance. The

TABLE 3. Volume-changes corresponding to stress component changes for uniqueness tests

Test No.	Elapsed time (min)	Stress component changes $\Delta\sigma$ (psi)	Δu_a (psi)	Δu_w (psi)	Percentage volume change Soil structure	Air	Water
Sample No. 15A							
C-1	1000	0	0	+10.54	−0.060	+0.080	−0.140
C-1	5000	0	0	+10.54	−0.188	+0.070	−0.258
C-2	1000	0	−9.96	0	+0.030	+0.190	−0.160
C-2	5000*	0	−9.96	–	+0.04	+0.44	−0.40
C-3	1000	−10.29	0	0	−0.110	−0.005	−0.105
C-3	5000	−10.29	0	0	−0.230	+0.280	−0.510
C-4	1000	0	0	+9.68	−0.050	+0.150	−0.200
C-4	5000	0	0	+9.68	−0.175	+0.565	−0.740
C-5	1000	0	−10.13	0	+0.045	+0.275	−0.230
C-5	5000	0	−10.13	0	+0.040	+0.860	−0.820
C-6	1000	−10.04	0	0	−0.110	+0.076	−0.185
C-6	5000	−10.04	0	0	−0.215	+0.385	−0.600
Sample No. 16							
C-7	2000	0	0	+9.95	+0.035	+0.163	−0.128
C-7	15000	0	0	+9.95	−0.090	+0.202	−0.292
C-8	2000	0	−10.03	0	+0.315	+0.375	−0.060
C-8	15000	0	−10.03	0	+0.290	+0.570	−0.280
C-9	2000	−9.88	0	0	−0.160	−0.055	−0.105
C-9	15000	−9.88	0	0	−0.255	+0.060	−0.315
C-10	2000	0	0	+9.88	−0.075	+0.040	−0.115
C-10	15000	0	0	+9.88	−0.220	+0.235	−0.455
C-11	2000	0	−9.96	0	+0.225	+0.315	−0.090
C-11	15000	0	−9.96	0	+0.265	+0.790	−0.525
C-12	2000	−9.94	0	0	−0.210	−0.083	−0.127
C-12	15000	−9.94	0	0	−0.430	+0.150	−0.580
C-13	2000	−9.85	0	0	−0.335	−0.182	−0.153
C-13	15000	−9.85	0	0	−0.705	−0.058	−0.647
Sample No. 17							
C-14	1000	0	0	+10.00	−0.100	+0.090	−0.190
C-14	5000	0	0	+10.00	−0.360	+0.230	−0.590
C-15	1000	0	−10.00	0	+0.018	+0.068	−0.050
C-15	5000	0	−10.00	0	−0.060	+0.230	−0.290
C-16	1000	−10.13	0	0	−0.140	−0.050	−0.090
C-16	5000	−10.13	0	0	−0.345	+0.135	−0.480
C-17	1000	0	0	+10.00	−0.095	+0.105	−0.200
C-17	5000	0	0	+10.00	−0.280	+0.320	−0.600
Sample No. 21							
C-18	2000	−9.80	0	0	−0.110	−0.078	−0.032
C-18	10000†	−9.80	0	0	−0.070	−0.080	+0.010
C-19	2000	0	−9.95	0	−0.015	−0.015	0.000
C-19	10000†	0	−9.95	0	−0.015*	−0.015	0.000*
C-20	2000	0	0	+10.04	−0.033	+0.167	−0.200
C-20	10000†	0	0	+10.04	−0.020*	+0.350	−0.370*
C-21	2000	−9.75	0	0	−0.290	−0.120	−0.170
C-21	10000†	−9.75	0	0	−0.320	−0.170	−0.150
C-22	2000	0	−9.86	0	−0.037	+0.053	−0.090
C-22	10000†	0	−9.86	0	−0.010*	+0.090	−0.100*
C-23	2000	0	0	+9.91	−0.127	+0.258	−0.385
C-23	10000†	0	0	+9.91	−0.180*	+0.310	−0.490*
C-24	2000	−9.68	0	0	−0.700	−0.405	−0.295
C-24	10000	−9.68	0	0	−0.820	−0.494	−0.326
C-25	2000	0	−10.03	0	+0.235	+0.805	−0.570
C-25	10000	0	−10.03	0	+0.180	+1.067	−0.887
C-26	2000	0	0	+4.87	−0.005	+0.655	−0.660

TABLE 3 (*continued*)

Test No.	Elapsed time (min)	Stress component changes $\Delta\sigma$ (psi)	Δu_a (psi)	Δu_a (psi)	Percentage volume change Soil structure	Air	Water
C-26	15000	0	0	+4.87	−0.010	+1.510	−1.520
C-27	2000	−9.74	0	0	−0.595	−0.130	−0.465
C-27	15000	−9.74	0	0	−0.730*	+0.120	−0.850*
C-28	2000	+9.30	0	0	+0.315	+9.260	+0.055
C-28	10000†	+9.30	0	0	+0.360*	+0.280	+0.080*
C-29	2000	0	0	−4.83	+0.090	−0.390	+0.480
C-29	20000†	0	0	−4.83	+0.260*	−0.670	+0.930*
C-30	2000	+29.20	0	0	+1.890	+0.870	+1.020
C-30	10000	+29.20	0	0	+2.160	+0.710	+1.450
C-31	2000	0	+30.03	0	−0.080	−0.730	+0.650
C-31	15000	0	+30.03	0	+0.045	−0.910	+0.955
C-32	2000	0	0	+29.86	−0.526	+0.456	−0.982
C-32	12000	0	0	+29.86	−0.567	+0.663	−1.230

*Deformations are very approximate.
†Estimated equilibrium conditions.

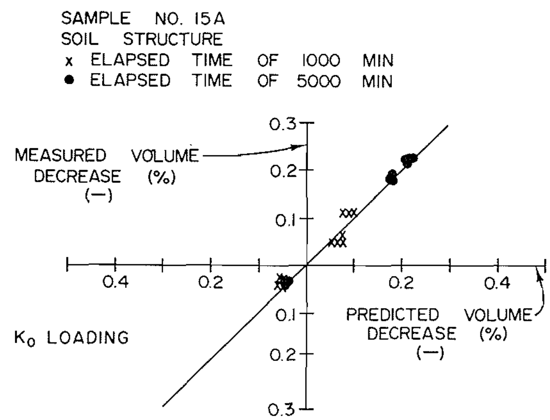

FIG. 6. Comparison of predicted and measured soil structure volume changes for sample No. 15A.

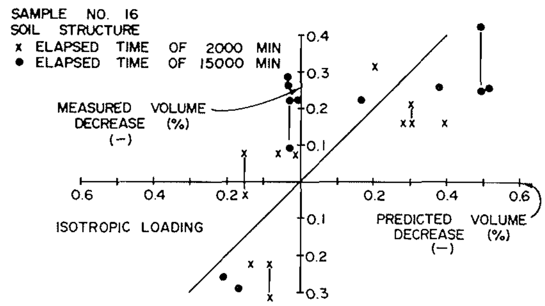

FIG. 7. Comparison of predicted and measured soil structure volume changes on sample No. 16.

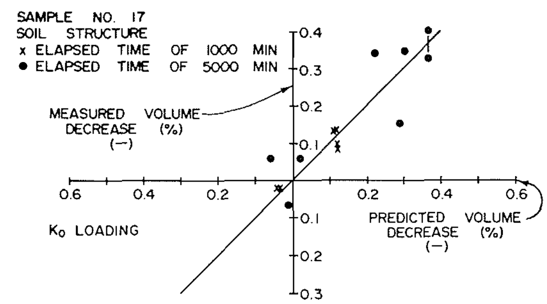

FIG. 8. Comparison of predicted and measured soil structure volume changes for sample No. 17.

weighted average intercept on the above samples was −0.001 and the slope was 0.82. The average results show the intercept tending towards zero and the slope tending towards one.

The correlation between the predicted and measured volume changes was also checked, assuming a non-linear constitutive surface (Table 5). The assumption made was that volume change varied as the logarithm of the stress state variables. The effect of a non-linear constitutive surface can be partly compensated for by correcting all slopes to the same stress point in space. Each compressibility is assumed to correspond to the mean of the initial and final stress points. All compressibilities are then corrected to the mean stress point.

Sample No. 15A showed a virtually 'perfect' correlation using the linear analysis and remains essentially the same for non-linear analysis. The non-linear analysis of sample No.

TABLE 4. Summary of regression analysis for soil structure assuming a linear constitutive surface

Sample No.	No. of comparisons	Critical coefficient of correlation 1% significance	Coefficient of correlation	Slope	Intercept
15A (1000 min)	12	0.68	0.961	0.901	−0.009
15A (5000 min)	12	0.68	0.999	0.993	−0.002
16 (2000 min)	12	0.68	0.750	0.678	+0.038
16 (15000 min)	12	0.68	0.779	0.763	+0.026
17 (1000 min)	6	0.91	0.961	0.850	−0.017
17 (5000 min)	6	0.91	0.904	0.650	−0.102
21 (2000 min)	30	0.46	0.293	0.230	−0.082
21 (Equilibrium conditions)	30	0.46	0.458	0.358	−0.090

TABLE 5. Summary of regression analysis using a logarithmic constitutive surface

Sample No.	No. of observations	Critical Coefficient of correlation 1% significance	Correlation coefficient	Slope	Intercept
15A (5000 min)	12	0.68	0.983	1.049	+0.014
16 (15000 min)	12	0.68	0.820	0.928	+0.041
17 (5000 min)	6	0.91	0.858	0.609	−0.111
21	30	0.46	0.410	0.339	−0.095
(Equilibrium conditions)	15*	0.62	0.390	0.293	−0.242

*Omission of first three and last four tests due to inconsistent order of changing stresses.

16 shows a marked improvement over the linear analysis. Sample No. 17 showed a slight decrease in correlation coefficient when the compressibilities were corrected for non-linearity. Sample No. 21 did not show any improvement in the statistical properties as a result of the correlations for non-linearity.

Tests No. 15A and 17 were performed on a heavily overconsolidated clay (*i.e.* Regina Clay). The effects of hysteresis should be small in the loading and unloading range considered. On the other hand, the kaolin sample compacted to one-half standard compaction, should have pronounced hysteresis.

The change in air pressure generally resulted in a reversed direction of soil structure deformation. In the case of the compacted kaolin (sample No. 21), there were considerably different amounts of volume change depending upon whether the volume was increasing or decreasing. It would appear that hysteresis associated with the change in the air pressure may be the key factor contributing to dispersion in the case of sample No. 21. The effect of hysteresis due to an air pressure change was investigated by assuming that the predicted volume change due to an air pressure change was equal to the measured volume change. The deformation due to a change in total and water pressures remain the same. Obviously, this will

TABLE 6. Summary of regression analysis using a non-linear constitutive surface and applying a correction for hysteresis

Sample No.	No. of observations	Critical Coefficient of correlation 1% significance	Correlation coefficient	Slope	Intercept
15A					
(5000 min)	12	0.68	0.969	0.971	−0.005
16	13	0.66	0.868	1.145	−0.006
(15000 min)	12*	0.68	0.934	0.953	−0.006
17	6	0.91	0.969	0.963	−0.008
(5000 min)	30	0.46	0.831	0.829	+0.034
21					
(Equilibrium					
conditions)	15†	0.62	0.939	0.957	−0.056

*Omission of C-10 and C-11 due to inaccurate measurements.
†Omission of first three and last four tests since the order of changing stresses does not lend itself to a systematic comparison of measured and predicted deformations.

produce an improvement in the correlation coefficient. However, conclusions must be based on the magnitude of the improvement of the correlation coefficient. Table 6 summarizes the statistical properties obtained when the above procedure is used to compensate for hysteresis.

The correlation coefficient for sample No. 16 increased slightly to 0.87. By omitting the data from the check using tests C-10 and C-11, the correlation coefficient improved to 0.93. Although there is some improvement in the correlation coefficient, it is not dramatic and may indicate that the Regina clay behavior approaches that of an elastic reversible material.

The statistical properties of sample No. 21 show a dramatic improvement as a result of compensating for hysteresis. Considering all the data, the correlation coefficient improved from 0.41 to 0.83 (non-linear analysis). The slope of the best-fit line increased from 0.34 to 0.83 and the intercept changed from −0.09 to +0.03. Omitting the results from the first three and the last four pressure increments, the statistics are further improved (Fig. 9). The correlation coefficient rose to 0.94, the slope to 0.96, while the intercept went to −0.06. The analysis isolates soil structure hysteresis as the prime cause of dispersion between predicted and measured deformations.

Water Phase

The laboratory testing technique associated with the water phase constitutive surface is considerably more difficult than for the soil structure constitutive surface. This factor influences the uniqueness analysis in addition to the effects of non-linearity and hysteresis.

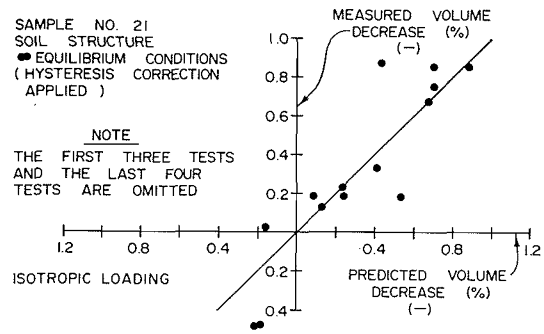

FIG. 9. Comparison of predicted and measured soil structure volume changes for sample No. 21 (hysteresis correction applied).

The statistical properties on the water phase volume changes deviate more from those of 'perfect' conditions than do the soil structure volume changes (Table 7). Figure 10 shows the comparison of predicted and measured water volume changes for sample No. 15A. This test was performed in a one-dimensional oedometer and it was later discovered that some moisture was lost from the sample by movement through the lucite walls of the chamber. The loss of even a small amount of moisture has a very pronounced affect on the uniqueness analysis. The excellent correlation from the uniqueness analysis on the soil structure (sample No. 15A) showed that hysteresis cannot be considered as the factor causing the

TABLE 7. Summary of regression analysis for water phase assuming a linear constitutive surface

Sample No.	No. of comparisons	Critical coefficient of correlation (1% significance)	Coefficient of correlation	Slope	Intercept
15A (1000 min)	12	0.68	0.484	0.124	−0.153
15A (5000 min)	12	0.68	0.514	0.116	−0.538
16 (2000 min)	12	0.68	0.645	0.168	−0.089
16 (15000 min)	12	0.68	0.153	0.027	−0.395
17 (1000 min)	6	0.91	0.583	2.028	+0.152
21 (2000 min)	30	0.46	0.582	0.606	−0.052
21 (Equilibrium conditions)	30	0.46	0.612	0.475	−0.227

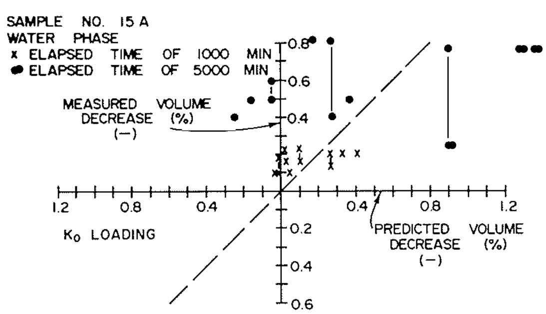

FIG. 10. Comparison of predicted and measured water phase volume changes for sample No. 15A.

low correlation for the water phase. It is quite definite that sample No. 15A experienced some loss of moisture during the test and this factor was investigated. An attempt was made to correct for the loss of moisture by observing the steady state rate of moisture loss at large elapsed times. When these corrections were applied, the correlation coefficient increased to 0.82 which is well above the critical correlation coefficient. However, the slope of the best-fit line is relatively low (0.55) and the intercept is quite far from zero (0.12).

Sample No. 16 also showed relatively poor correlation between the predicted and measured volume changes. When the results were corrected for the effects of soil structure hysteresis, the correlation coefficient increased from 0.15 to 0.89. This is well above the critical correlation coefficient of 0.71 for a 1% level of significance. The slope increased from 0.43 to 0.80. The marked improvement in statistical properties indicates that hysteresis in the soil structure produced dispersion in the water phase constitutive surface.

The results from sample No. 17 showed a close agreement between the predicted and measured water volume changes. However, the total number of checks was only six. Sample No. 21 (Fig. 11) showed relatively good agreement between the predicted and measured results. When the results are corrected for hysteresis (and the first three and the last four checks are omitted), the correlation coefficient increased to 0.82 which is well above the

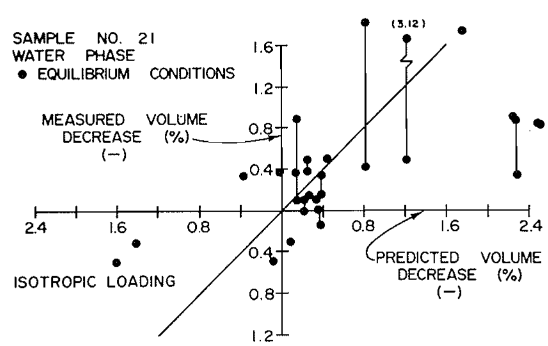

FIG. 11. Comparison of predicted and measured water phase volume changes for sample No. 21 (equilibrium conditions).

critical correlation coefficient of 0.46 for a 1% level of significance. The slope of the best-fit line increased to 0.54. The intercept was −0.63 which is higher than desirable but appears related to two 'checks' showing a large discrepancy.

Conclusions

1. Two consecutive relations are required to describe the volume change behavior of an unsaturated soil. The first proposed equation is for the soil structure.

$$\epsilon = \frac{1}{v}\cdot\frac{\partial v}{\partial(\sigma - u_a)}\cdot d(\sigma - u_a) + \frac{1}{v}\cdot\frac{\partial v}{\partial(u_a - u_w)}\cdot d(u_a - u_w)$$

The second proposed equation is for the change in volume of water in the element.

$$\theta_w = \frac{1}{v}\cdot\frac{\partial v}{\partial(\sigma - u_a)}\cdot d(\sigma - u_a) + \frac{1}{v}\cdot\frac{\partial v}{\partial(u_a - u_w)}\cdot d(u_a - u_w)$$

The proposed constitutive equations were tested for uniqueness by determining the modulus of deformation in two directions and comparing the predicted and measured volume changes in a third direction.

2. Tests were performed on samples of overconsolidated, undisturbed Regina Clay and compacted kaolin. These soils are typical of the range of properties commonly encountered.

(a) For the soil structure constitutive relationship, the volume changes predicted agreed well with the measured values for Regina Clay. Corrections for non-linearity of the constitutive surface and hysteresis upon loading and unloading showed further improvement in the correlation. The compacted kaolin showed relatively poor correlation between the predicted and measured volume changes. However, the correlation showed a dramatic improvement when corrections were applied for hysteresis.

Since hysteresis is a significant factor contributing to non-uniqueness, it is concluded that the proposed constitutive equation for the soil structure be used for monotonic deformations. Therefore, the compressibility moduli would be different for decreasing volume changes than for volume increases.

(b) The agreement between the predicted and measured water phase volume changes were not as close as those for the soil structure. The main cause of dispersion is associated with the difficulty in measuring minute water volume changes over long periods of time. This factor was in addition to the effects of non-linearity and hysteresis. Attempted corrections for the above factors showed a marked improvement in the correlation of predicted and measured water volume changes. It is concluded that the proposed water phase constitutive equation be used as well for monotonic water phase volume changes.

3. For complete monitoring of unsaturated soil behavior, laboratory testing should be done under conditions where the total, air and water pressures are controlled. The overall volume change should be measured.

4. The laboratory tests on unsaturated soils indicate that the proposed constitutive equations can be used for engineering practice.

AITCHISON, G. D. 1967. Separate roles of site investigation, quantification of soil properties, and selection of operational environment in the determination of foundation design on expansive soils. Proc. 3rd Asian Reg. Conf. Soil Mech. Found. Eng. vol. 2, Haifa, Isr.

——— 1969. Soil suction in foundation design. Proc. 7th Int. Conf. Soil Mech. Found. Eng., Mexico, vol. 2, pp. 1–8.

BARDEN, L., MADEDOR, A. O., and SIDES, G. R. 1969. Volume change characteristics of unsaturated clay. ASCE, J. Soil Mech. Found. Div. **95**(SM1), pp. 33–51.

BIOT, M. A. 1941. General theory of three dimensional consolidation. J. Appl. Phys. **12**, pp. 155–164.

BISHOP, A. W. and BLIGHT, G. E. 1963. Some aspects of effective stress in saturated and unsaturated soils. Geotechnique, **13**, pp. 177–197.

BRACKLEY, I. J. A. 1971. Partial collapse in unsaturated expansive clay. Proc. 5th Regional Conf. Soil Mech. Found. Eng., South Africa, pp. 23–30.

BURLAND, J. B. 1965. Some aspects of the mechanical behavior of partly saturated soils. *In*: Moisture equilibrium and moisture changes in soils beneath covered areas. Butterworth and Company (Australia) Ltd., Sydney, Aust.

CHILDS, E. C. 1969. The physical basis of soil water phenomena. Wiley-Interscience, John Wiley and Sons Ltd., London, Engl.

COLEMAN, J. D. 1962. Stress/strain relations for partly saturated soil. Corresp. Geotechnique, **12**(4), pp. 348–350.

DAVIES, J. T. and RIDEAL, E. K. 1963. Interfacial phenomena. 2nd ed., Academic Press Inc., New York, N.Y.

FREDLUND, D. G. 1973*a*. Volume change behavior of unsaturated soils. PhD Diss., Univ. Alberta, Edmonton, Alta.

——— 1973*b*. Discussion on the 2nd Technical Session Division Two (Flow shear Strength). Proc. 3rd Int. Conf. on Expansive Soils, vol. II, Jerusalem Academic Press Ltd., Haifa, Isr., pp. 71–76.

——— 1974. Engineering approach to soil continua. Proc. 2nd Symp. on the Applications of Solid Mechanics, vol. 1, pp. 46–59, Hamilton, Ont.

——— 1975. A diffused air volume indicator for unsaturated soils. Can. Geotech. J., **12**(4), pp. 533–539.

JENNINGS, J. E. and BURLAND, J. B. 1962. Limitations to the use of effective stresses in partly saturated soils. Geotechnique, **12**(2), pp. 125–144.

MATYAS, E. L. and RADHAKRISHNA, H. S. 1968. Volume change characteristics of partially saturated soils. Geotechnique, **18**(4), pp. 432–448.

NEVILLE, A. M. and KENNEDY, J. B. 1966. Basic statistical methods for engineers and scientists. Int. Textbook Co., Scranton, Penn.