BIOGRAPHY

D.G. Fredlund obtained his bachelor of science degree in civil engineering from the University of Saskatchewan, Saskatoon and his master of science degree in 1964 from the University of Alberta, Edmonton. During the summers 1962 and 1963, he worked with the Division of Building Research, National Research Council of Canada, Saskatoon. From 1964 to 1966 he was employed by R.M. Hardy and Associates Limited, Edmonton. In 1966 he accepted a teaching position in civil engineering at the University of Saskatchewan where he remains to the present. From 1970 to 1972 he completed his doctor of philosophy degree at the University of Alberta. In addition to teaching and research, he is a staff consultant to Ground Engineering Limited, Regina. His research has been primarily directed towards the areas of slope stability and the behavior of unsaturated soils.

BIOGRAPHY

John Krahn graduated with his Bachelor of Science degree in Civil Engineering in 1969 followed by his Master of Science degree in 1970. Both degrees were granted by the University of Saskatchewan in Saskatoon.

John went on to earn his Ph.D. degree from the University of Alberta in 1974 and upon graduation, he accepted a position with EBA Engineering Consultants Ltd. in Edmonton. While with EBA, John contributed largely to a feasibility study of a proposed major earth dam in Central Alberta and was involved with the open pit mining operation in the Athabasca tar sands.

In the fall of 1975, John accepted a teaching position with the University of Saskatchewan in Saskatoon. In addition to his University responsibilities John is a staff consultant to Ground Engineering Ltd., in Regina.

COMPARISON OF SLOPE STABILITY METHODS OF ANALYSIS

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Abstract

The paper compares six methods of slices commonly used for slope stability analysis. The factor of safety equations are written in the same form, recognizing whether moment and/ or force equilibrium is explicitly satisfied. The normal force equation is of the same form for all methods with the exception of the Ordinary method. The method of handling the interslice forces differentiates the normal force equations.

A new derivation for the Morgenstern-Price method is presented and is called the 'Best-Fit Regression' solution. It involves independently solving for the force and moment equilibrium factors of safety for various lambda values. The 'Best-Fit Regression' solution gives the same factor of safety as the 'Newton-Raphson' solution. The 'Best-Fit Regression' solution is readily comprehended, giving a complete understanding of the variation of factor of safety with lambda.

COMPARAISON DES METHODES D'ANALYSE DE STABILITE DE PENTE

Par: D.G. Fredlund et J. Krahn

SOMMAIRE DU CONTENU (Traduction professionnelle)

Cet article compare six méthodes d'analyses par tranches généralement employées pour les études de stabilité de pente. Les équations du facteur de sécurité sont écrites sous la même forme, en précisant si c'est l'équilibre des forces ou des moments qui est satisfait. L'équation de la force normale s'exprime de la Même façon pour toutes les méthodes sauf pour le cas des méthodes ordinaires des tranches. C'est la façon de manipuler les forces qui s'exercent entre les tranches qui différencie les équations de la force normale.

Une nouvelle dérivation de la méthode de Morgestern-Price est présentée sous le nom de "Solution de la meilleure régression". La procédure engendrée résoud des facteurs de sécurité indépendants pour l'équilibre des forces des moments et cela pour différentes valeurs de lambda. Cette nouvelle méthode donne le même facteur de sécurité que la méthode de "Newton-Raphson". La méthode de la meilleure régression est rapidement assimilée et permet d'obtenir une bonne compréhension de la variation du facteur de sécurité en fonction de lambda.

COMPARISON OF SLOPE STABILITY METHODS OF ANALYSIS

D.G. Fredlund and J. Krahn

University of Saskatchewan

Introduction

The geotechnical engineer frequently uses limit equilibrium methods of analysis when studying slope stability problems. The methods of slices have become the most common method due to their ability to accommodate complex geometrics, and variable soil and water pressure conditions (Terzaghi and Peck, 1967). During the past three decades approximately one dozen methods of slices have been developed (Wright, 1969). They differ in (i) the statics employed in deriving the factor of safety equation and (ii) the assumption used to render the problem determinate (Fredlund, 1975).

This paper is primarily concerned with six of the most commonly used methods:

(i) Ordinary or Fellenius Method (Sometimes referred to as the Swedish Circle method of the Conventional method).

(ii) Simplified Bishop Method

(iii) Spencer's Method

(iv) Janbu's Simplified Method

(v) Janub's Rigorous Method

(vi) Morgenstern-Price Method

The objectives of this paper are:

(1) to compare the various methods of slices in terms of consistent procedures for deriving the factor of safety equations. All equations are extended to the case of a composite

failure surface and also consider partial submergence, line loadings and earthquake loadings.

(2) to present a new derivation for the Morgenstern-Price method. The proposed derivation is more consistent with that used for the other methods of analysis but utilizes the elements of statics and the assumption proposed by Morgenstern and Price (1965). The Newton-Raphson numerical technique is not used to compute the factor of safety and lambda.

(3) to compare the factors of safety obtained by each of the methods for several example problems. The University of Saskatchewan 'SLOPE' computer program was used for all computer analyses (Fredlund, 1974).

(4) to compare the relative computational costs involved in using the various methods of analysis.

Definition of Problem

Figure 1 shows the forces that must be defined for a general slope stability problem. The variables associated with each slice are defined as follows:

W = total weight of the slice of width, b, and height, h.

P = total normal force on the base of the slice over a length, l.

S_m = shear force mobilized on the base of the slice. It is a percentage of the shear strength as defined by the Mohr-Coulomb equation. That is,

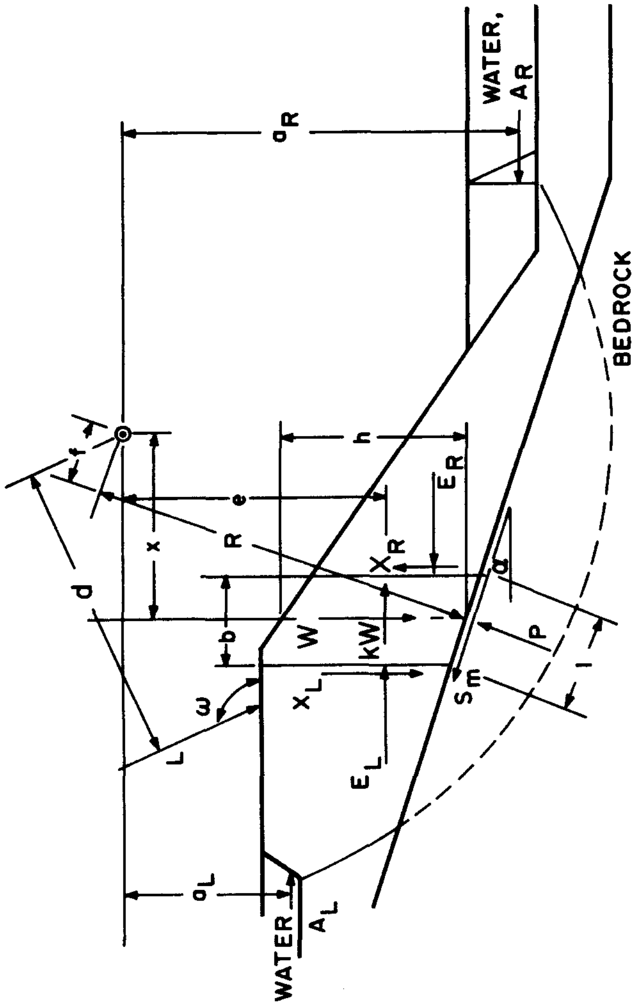


FIG. I FORCES ACTING FOR THE METHOD OF SLICES APPLIED TO A COMPOSITE SLIDING SURFACE

$S_m = \{c' + (\frac{P}{l} - u)\cdot\tan \phi'\}/F$

where c' = effective cohesion parameter

ϕ' = effective angle of internal friction

F = factor of safety

u = porewater pressure

R = Radius or the moment arm associated with the mobilized shear force, S_m

f = perpendicualr offset of the normal force from the center of rotation.

x = horizontal distance of the slice from the center of rotation.

α = angle between the tangent to the center of the base of each slice and the horizontal

E = horizontal interslice forces. The 'L' and 'R' subscripts designate the left and right sides of the slice, respectively.

X = vertical interslice forces. The left and right sides are designated by 'L' and 'R' subscripts, respectively.

k = seismic coefficient to account for a dynamic horizontal force.

e = vertical distance from the centroid of each slice to to center of rotation.

A uniform load on the surface can be taken into account as a soil layer of suitable unit weight and density. The following variables are required to define a line load:

L = magnitude of the line load (force per unit width)

ω = angle of the line load from the horizontal.

d = perpendicular distance from the line load to the center of rotation.

The effect of partial submergence of the slope or tension cracks with water require additional variables to be defined:

A = resultant water forces

a = perpendicular distance from the resultant water force to the center of rotation. The 'L' and 'R' subscripts designate the left and right sides of the slope, respectively.

Derivations for Factor of Safety

The elements of statics that can be used to derive the factor of safety are summations of forces in two directions and the summation of moments. These along with the failure criteria, are insufficient to make the problem determinate. More information must be known about either the normal force distribution or the interslice force distribution. Either additional elements of physics or an assumption must be invoked to render the problem determinate. All methods considered in this paper use the latter procedure, each assumption giving rise to a different method of analysis. For comparison purposes, each equation is derived using a consistent utilization of the equations of statics.

Ordinary or Fellenius Method

The Ordinary method is considered the simplest of the methods of slices since it is the only procedure that results in a linear factor of safety equation. It is generally stated that the interslice forces can be neglected because they are parallel to the base of each slice (Fellenius, 1936).

However, Newton's principle of 'action equals reaction' is not satisfied between slices (Figure 2). The indescriminate change in direction of the resultant interslice force from one slice to the next results in factor of safety errors that may be as much as 60 percent (Whitman and Bialey, 1967).

The normal force on the base of each slice is derived either from summation of forces perpendicular to the base or from the summation of forces in a vertical and horizontal direction.

[1] $\Sigma F_V = 0; \quad W - P\cdot\cos\alpha - S_m\cdot\sin\alpha = 0$

[2] $\Sigma F_H = 0; \quad S_m\cdot\cos\alpha - P\cdot\sin\alpha + k\cdot W = 0$

Substituting equation [2] into [1] and solving for the normal force gives,

[3] $P = W\cdot\cos\alpha - k\cdot W\cdot\sin\alpha$

The factor of safety is derived from the summation of moments about a common point (i.e. either a fictitious or real center of rotation for the entire mass).

[4] $\Sigma M_o = 0; \quad \Sigma W\cdot x - \Sigma S_m\cdot R - \Sigma P\cdot f + \Sigma k\cdot W\cdot e \pm A\cdot a + L\cdot d = 0$

Introducing the failure criteria and the normal force, [3], and solving for the factor of safety gives,

[5] $F = \dfrac{\Sigma\{c'\cdot l\cdot R + (P - u\cdot l)\cdot R\cdot\tan\phi'\}}{\Sigma W\cdot x - \Sigma P\cdot f + \Sigma k\cdot W\cdot e \pm A\cdot a + L\cdot d}$

Simplified Bishop Method

The Simplified Bishop method neglects the interslice shear forces and thus assumes that a normal or horizontal force adequately defines the interslice forces (Bishop, 1955). The normal force on the base of each slice is derived by summing forces in a vertical direction (same as equation [1]). Substituting in the failure cirteria and solving for the normal force gives,

$$[6] \quad P = \frac{W - \frac{c' \cdot l}{F} + \frac{u \cdot l \cdot \tan \phi' \cdot \sin \alpha}{F}}{m_\alpha}$$

where $m_\alpha = \cos \alpha + \frac{\sin \alpha \cdot \tan \phi'}{F}$

The factor of safety is derived from the summation of moments about a common point. This equation is the same as equation [4] since the interslice forces cancel out. Therefore, the factor of safety equation is the same as for the Ordinary method (equation [5]). However, the definition of the normal force is different.

Spencer's Method

Spencer's method assumes there is a constant relationship between the magnitude of the interslice shear and normal forces (Spencer 1967).

$$[7] \quad \tan \theta = \frac{X_L}{E_L} = \frac{X_R}{E_R}$$

where θ = angle of the resultant interslice force from the horizontal.

Spencer (1967) summed forces perpendicular to the interslice forces to derive the normal force. The same result can be obtained by summing forces in a vertical and horizontal direction.

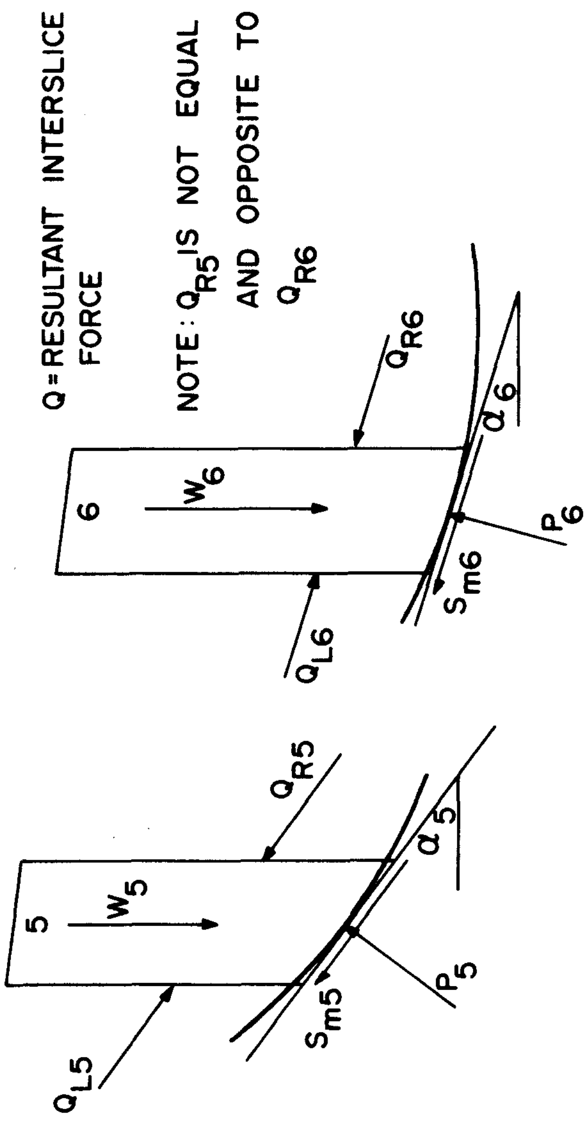


FIG. 2 INTERSLICE FORCES FOR THE ORDINARY METHOD

[8] $\Sigma F_V = 0; \quad W - (X_R - X_L) - P \cdot \cos\alpha - S_m \cdot \sin\alpha = 0$

[9] $\Sigma F_H = 0; \quad -(E_R - E_L) + P \cdot \sin\alpha - S_m \cdot \cos\alpha + k \cdot W = 0$

The normal force can be derived from equation [8] and then the horizontal interslice force is obtained from equation [9].

[10] $$P = \frac{W - (E_R - E_L) \cdot \tan\theta - \frac{c' \cdot l \cdot \sin\alpha}{F} + \frac{u \cdot l \cdot \tan\phi' \cdot \sin\alpha}{F}}{m_\alpha}$$

Spencer (loc cit) derived two factor of safety equations. One is based on the summation of moments about a common point and the other on the summation of forces in a direction parallel to the interslice forces. The moment equation is the same as for the Ordinary and Simplified Bishop method (i.e. equation [4]). The factor of safety equation is the same as [5].

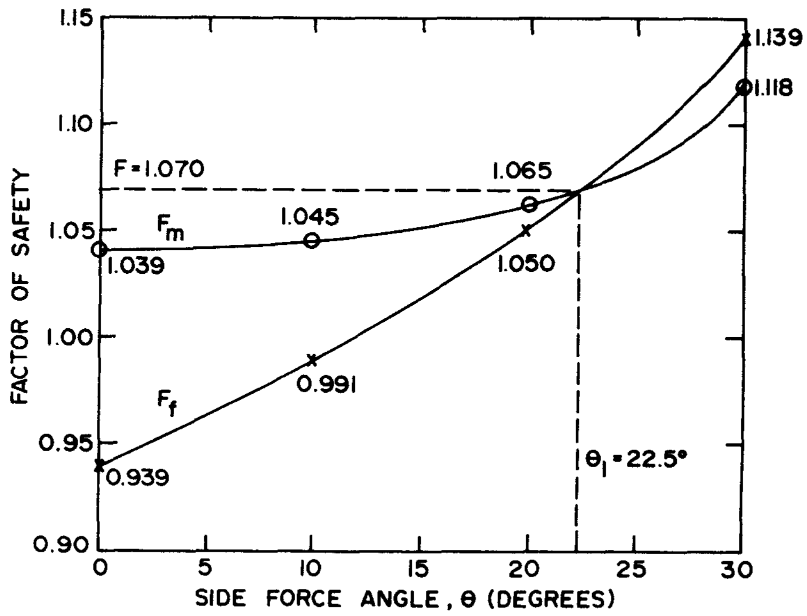
The factor of safety equation based on force equilibrium can also be derived by summing forces in a horizontal direction.

[11] $\Sigma F_H = 0; \quad \Sigma(E_L - E_R) + \Sigma P \cdot \sin\alpha - \Sigma S_m \cdot \cos\alpha + \Sigma k \cdot W \pm A - L \cdot \cos\omega = 0$

The interslice forces ($E_L - E_R$), must cancel out and the factor of safety equation with respect to force equilibrium reduces to,

[12] $$F_f = \frac{\Sigma\{c' \cdot l \cdot \cos\alpha + (P - u \cdot l) \cdot \tan\phi' \cdot \cos\alpha\}}{\Sigma P \cdot \sin\alpha + \Sigma k \cdot W \pm A - L \cdot \cos\omega}$$

Spencer's method yields two factors of safety for each angle of side forces. However, at some angle of the interslice forces, the two factors of safety are equal (Figure 3) and both moment and force equilibrium are satisfied.



SOIL PROPERTIES

$c'/\gamma h = 0.02$
$\phi' = 40°$
$r_u = 0.5$

GEOMETRY

Slope = 26.5°
Height = 100 Ft.

FIG. 3 VARIATION OF THE FACTOR OF SAFETY WITH RESPECT TO MOMENT AND FORCE EQUILIBRIUM VERSUS THE ANGLE OF THE SIDE FORCES (FROM SPENCER, 1967)

The Corps of Engineers method, sometimes referred to as Taylor's modified Swedish method, is equivalent to the force equilibrium portion of Spencer's method when the interslice forces are set equal to the average surface slope.

Janbu's Simplified Method

Janbu's simplified method uses a correction factor, f_o, to account for the effect of the interslice shear forces. The correction factor is related to cohesion, angle of internal friction and the shape of the failure surface (Janbu et al, 1956). The normal force is derived from the summation of vertical forces (equation [8]); with the interslice shear forces ignored.

$$[13] \quad P = \frac{W - \frac{c'\cdot l\cdot \sin\alpha}{F} + \frac{u\cdot l\cdot \tan\phi'\cdot \sin\alpha}{F}}{m_\alpha}$$

The horizontal force equilibrium equation is used to derive the factor of safety (i.e. equation [11]). The sum of the inter-slice forces must cancel and the factor of safety equation becomes,

$$[14] \quad F_o = \frac{\Sigma\{c'\cdot l\cdot \cos\alpha + (P-u\cdot l)\tan\phi'\cdot \cos\alpha\}}{\Sigma P\cdot \sin\alpha + \Sigma k\cdot W \pm A-L\cdot \cos\omega}$$

F_o is used to designate the factor of safety uncorrected for the interslice shear forces. The corrected factor of safety is,

$$[16] \quad F = f_o\cdot F_o$$

Janbu's Rigorous Method

Janbu's Rigorous method assumes that the point at which the interslice forces act can be defined by a "line of thrust".

New terms used are defined as follows: (See Figure 4)

t_L, t_R = vertical distance from the base of the slice to the 'line of thrust' on the left and right sides of the slice, respectively.

α_t = angle between the 'line of thrust' on the right side of a slice and the horizontal.

The normal force on the base of the slice is derived from the summation of vertical forces.

$$[16] \quad P = \frac{W-(X_R-X_L)-\frac{c'\cdot l\cdot \sin\alpha}{F} + \frac{u\cdot l\cdot \tan\phi'\sin\alpha}{F}}{m_\alpha}$$

The factor of safety equation is derived from the summation of horizontal forces (i.e. equation [11]). Janbu's rigorous analysis differs from the simplified analysis in that the shear forces are kept in the derivation of the normal force. The factor of safety equation is the same as Spencer's equation based on force equilibrium (i.e. equation [12]).

In order to solve the factor of safety equation, the interslice shear forces must be evaluated. For the first iteration, the shears are set to zero. For subsequent interations, the interslice forces are computed from the sum of the moments about the center of the base of each slice.

$$[17] \quad \Sigma M_c = 0; \quad X_L\cdot b+(X_R-X_L)\cdot\frac{b}{2}+(E_R-E_L)\cdot(t_L+\frac{b}{2}\cdot\tan\alpha-b\cdot\tan\alpha_t) -E_L\cdot b\cdot\tan\alpha_t-k\cdot W\cdot\frac{h}{2}$$

Several terms become negligible as the width of the slice, b, is reduced to a width, dx. These terms are $(X_R-X_L)\cdot\frac{b}{2}$, $(E_R-E_L)\cdot\frac{b}{2}\cdot\tan\alpha$ and $(E_R-E_L)\cdot b\cdot\tan\alpha_t$. Eliminating these terms and dividing by the slice width, the shear force on the right

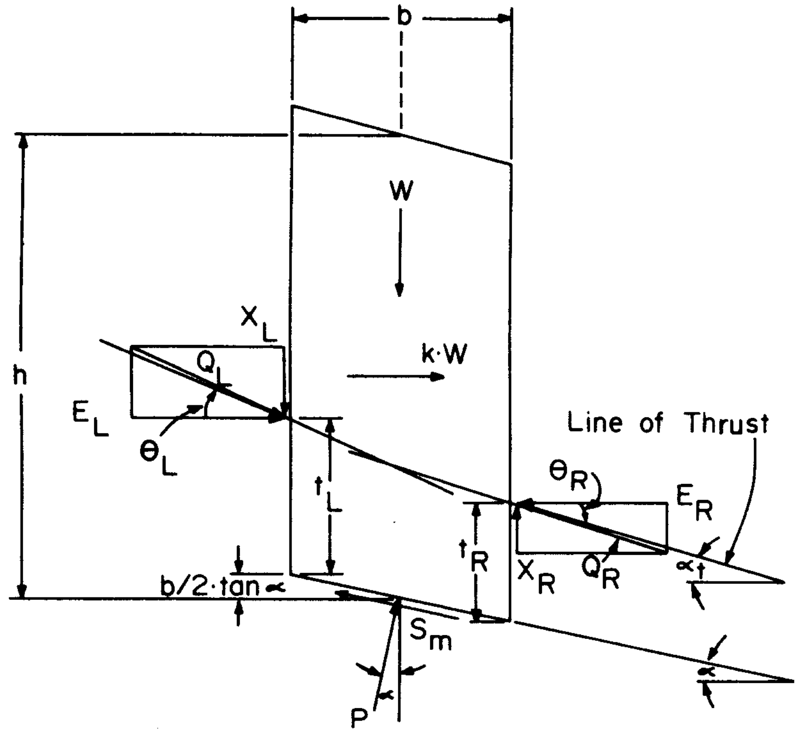


FIG. 4 FORCES ACTING ON EACH SLICE FOR JANBU'S RIGOROUS METHOD

side of a slice is,

$$[18] \quad X_R = E_R \cdot \tan\alpha_t - (E_R - E_L) \cdot \frac{t_L}{b} + k \cdot W \cdot \frac{h}{2}$$

The horizontal interslice forces, required for solving equation [18], are obtained by combining the summation of vertical and horizontal forces on each slice.

$$[19] \quad (E_R - E_L) = [W - (X_R - X_L)] \cdot \tan\alpha - \frac{S_m}{\cos\alpha} + k \cdot W$$

The horizontal interslice forces are obtained by integration from left to right across the slope. The magnitude of the interslice shear forces in equation [19] lag by one iteration. Each iteration gives a new set of shear forces.

Morgenstern-Price Method

The Morgenstern-Price method assumes an arbitrary mathematical function to describe the direction of the interslice forces.

$$[20] \quad \lambda \cdot f(x) = X/E$$

where λ = (Lambda) a constant to be evaluated when solving for the factor of safety.

$f(x)$ = functional variation with respect to x.

Figure 5 shows typical functions (i.e. f(x)). When the function is a constant, the Morgenstern-Price method is the same as the Spencer method. Figure 6 shows how the half sine function and lambda are used to designate the direction of the interslice forces.

Morgenstern and Price (1965) based their solution on the summation of tangential and normal forces to each slice.

The force equilibrium equations were combined and then the Newton-Raphson numerical technique was used to solve the moment and force equations for the factor of safety and lambda.

In this paper, an alternate derivation for the Morgenstern-Price method is proposed. The solution satisfies the same elements of statics but the derivation is more consistent with that used in the other methods of slices. It also presents a complete description of the variation of factor of safety with respect to lambda.

The normal force is derived from the vertical force equilibrium equation (Equation [16]). Two factor of safety equations are computed; one with respect to moment equilibrium and one with respect to force equilibrium. The moment equilibrium equation is taken with respect to a common point. Even if the sliding surface is composite, a fictitious common center can be used. The equation is the same as that obtained for the Ordinary method, Simplified Bishop method and Spencer's method (Equations [4] and [5]). The factor of safety with respect to force equilibrium is the same as that derived for Spencer's Method (Equation [12]). The interslice shear forces are computed in a manner similar to that presented for Janbu's Rigorous method. On the first iteration, the vertical shear forces are set to zero. One subsequent iterations, the horizontal interslice forces are first computed (Equation [19]) and then the vertical shear forces are computed using an assumed lambda value and side force function.

[21] $X_R = E_R \cdot \lambda \cdot f(x)$

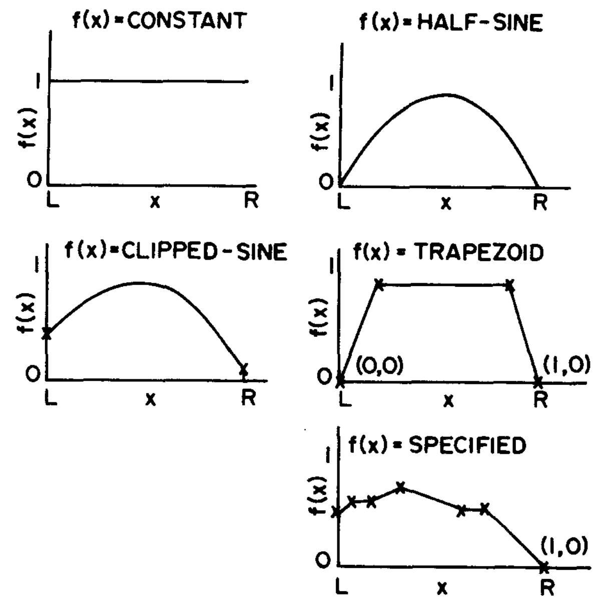


FIG. 5 FUNCTIONAL VARIATION OF THE DIRECTION OF THE SIDE FORCE WITH RESPECT TO THE X DIRECTION

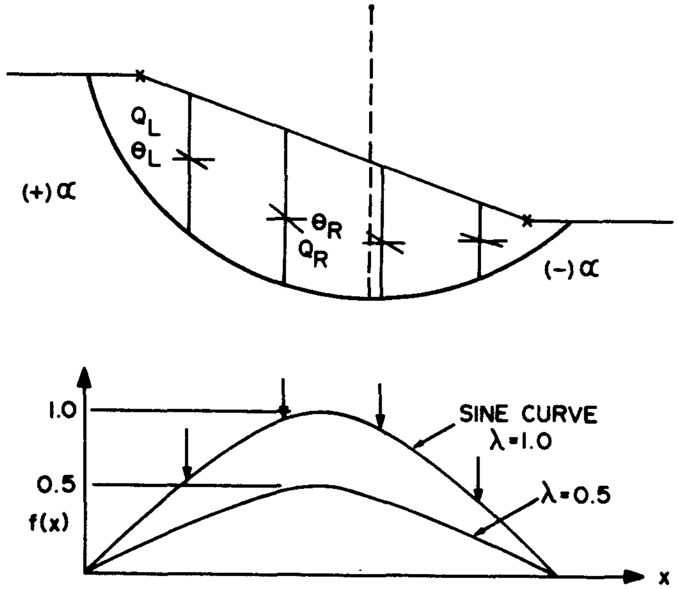


FIG. 6 SIDE FORCE DESIGNATION FOR THE MORGENSTERN - PRICE METHOD

The side forces are recomputed for each iteration. The moment and force equilibrium factors of safety are solved for a range of lambda values and a specified side force function. These factors of safety are plotted in a manner similar to that used for Spencer's method (Figure 7). The factors of safety versus lambda are fit by a second order polynomial regression and the point of intersection satisfies both force and moment equilibrium.

Comparison of Methods of Analysis

All of the methods of slices satisfying overall moment equilibrium can be written in the same form.

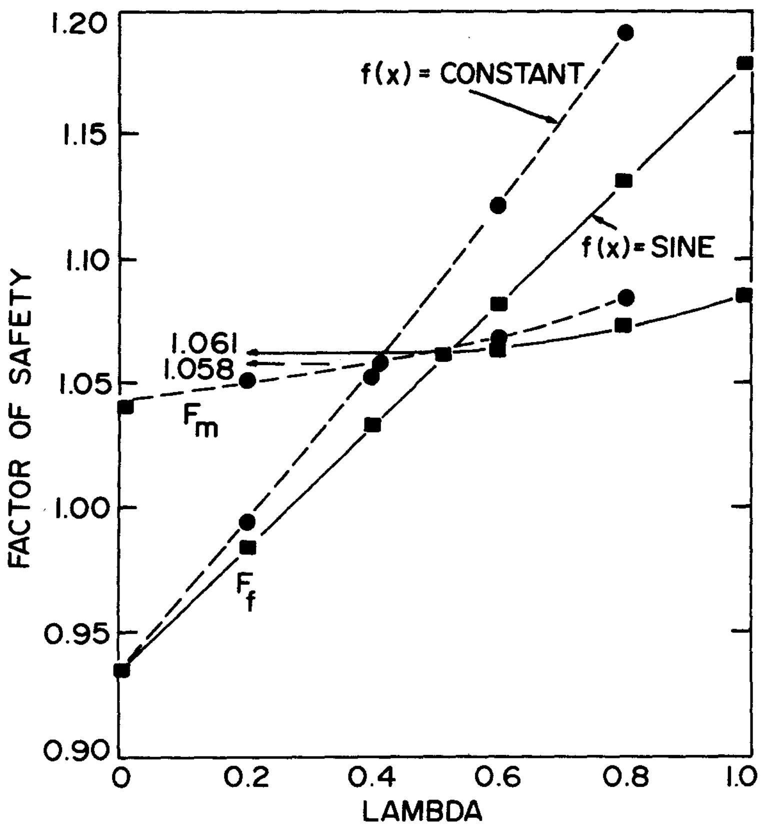
$$[22] \quad F_m = \frac{\Sigma c' \cdot l \cdot R + \Sigma(P - u \cdot l) \cdot R \cdot \tan \phi'}{\Sigma W \cdot x - \Sigma k \cdot W \cdot e \pm A \cdot a + L \cdot d}$$

All methods satisfying overall force equilibrium have the following form for the factor of safety equation.

$$[23] \quad F_f = \frac{\Sigma c' \cdot l \cdot \cos\alpha + \Sigma(P - u \cdot l) \cdot \tan \phi' \cdot \cos\alpha}{\Sigma P \cdot \sin\alpha + \Sigma k \cdot W \pm A - L \cdot \cos\omega}$$

The factor of safety equations can be visualized as consisting of the following components:

	Moment Equilibrium	Force Equilibrium
Cohesion	$\Sigma c' \cdot l \cdot R$	$\Sigma c' \cdot l \cdot \cos\alpha$
Friction	$\Sigma(P-u \cdot l) \cdot R \cdot \tan\phi'$	$\Sigma(P-u \cdot l) \cdot \tan\phi' \cdot \cos\alpha$
Weight	$\Sigma W \cdot x$	---
Normal	$\Sigma P \cdot f$	$\Sigma P \cdot \sin\alpha$
Earthquake	$\Sigma k \cdot W \cdot e$	$\Sigma k \cdot W$
Partial Submergence	$A \cdot a$	A
Line Loading	$L \cdot d$	$L \cdot \cos\omega$



SOIL PROPERTIES

$\frac{c'}{\gamma h} = 0.02$

$\phi' = 40°$

$r_u = 0.5$

GEOMETRY

SLOPE = 26.5°

HEIGHT = 100 FT.

FIG. 7 VARIATION OF FACTOR OF SAFETY WITH RESPECT TO MOMENT AND FORCE EQUILIBRIUM VERSUS LAMBDA FOR THE MORGENSTERN-PRICE METHOD

From a theoretical standpoint, the derived factor of safety equations differ in i) the equations of statics satisfied explicitly for the overall slope and ii) the assumption to make the problem determinate. The assumption used changes the evaluation of the interslice forces in the normal force equation (Table 1). All methods, with the exception of the Ordinary method, have the same form of equation for the normal force.

$$P = \frac{W - (X_R - X_L) - \frac{c' \cdot l \cdot \sin\alpha}{F} + \frac{u \cdot l \cdot \tan\phi' \cdot \sin\alpha}{F}}{m_\alpha} \qquad [24]$$

where $m_\alpha = \cos\alpha + \frac{\sin\alpha \cdot \tan\phi'}{F}$

It is possible to view the analytical aspects of slope stability in terms of one factor of safety equation satisfying overall moment equilibrium and another satisfying overall force equilibrium. Then each method becomes a special case of the 'Best-Fit Regression' solution to the Morgenstern-Price method.

Figure 8 shows an example problem involving both circular and composite failure surfaces. The results of six possible combinations of geometry, soil properties and water conditions are presented in Table II. This is not meant to be a complete study of the quantitative relationship between various methods but rather a typical example.

The various methods (with the exception of the Ordinary method), can be compared by plotting factor of safety versus

TABLE I

Comparison of Factor of Safety Equations

Method	Factor of Safety Based On: Moment Equilibrium	Factor of Safety Based On: Force Equilibrium	Normal Force Equation
Ordinary or Fellenius	x		$P = W\cdot\cos\alpha - k\cdot W\cdot\sin\alpha$
Simplified Bishop	x		$P = \dfrac{W - \dfrac{c^{1}\cdot l}{F} + \dfrac{u\cdot l\cdot\tan\phi^{1}\cdot\sin\alpha}{F}}{m_\alpha}$
Spencer's	x	x	$P = \dfrac{W - (X_R - X_L) - \dfrac{c^{1}\cdot l\cdot\sin\alpha}{F} + \dfrac{u\cdot l\cdot\tan\phi^{1}\cdot\sin\alpha}{F}}{m_\alpha}$
Janbu's Simplified		x	$P = \dfrac{W - \dfrac{c^{1}\cdot l\cdot\sin\alpha}{F} + \dfrac{u\cdot l\cdot\tan\phi^{1}\cdot\sin\alpha}{F}}{m_\alpha}$
Janbu's Rigorous		x	$P = \dfrac{W - (X_R - X_L) - \dfrac{c^{1}\cdot l\cdot\sin\alpha}{F} + \dfrac{u\cdot l\cdot\tan\phi^{1}\cdot\sin\alpha}{F}}{m_\alpha}$
Morgenstern-Price	x	x	$P = \dfrac{W - (X_R - X_L) - \dfrac{c^{1}\cdot l\cdot\sin\alpha}{F} + \dfrac{u\cdot l\cdot\tan\phi^{1}\cdot\sin\alpha}{F}}{m_\alpha}$

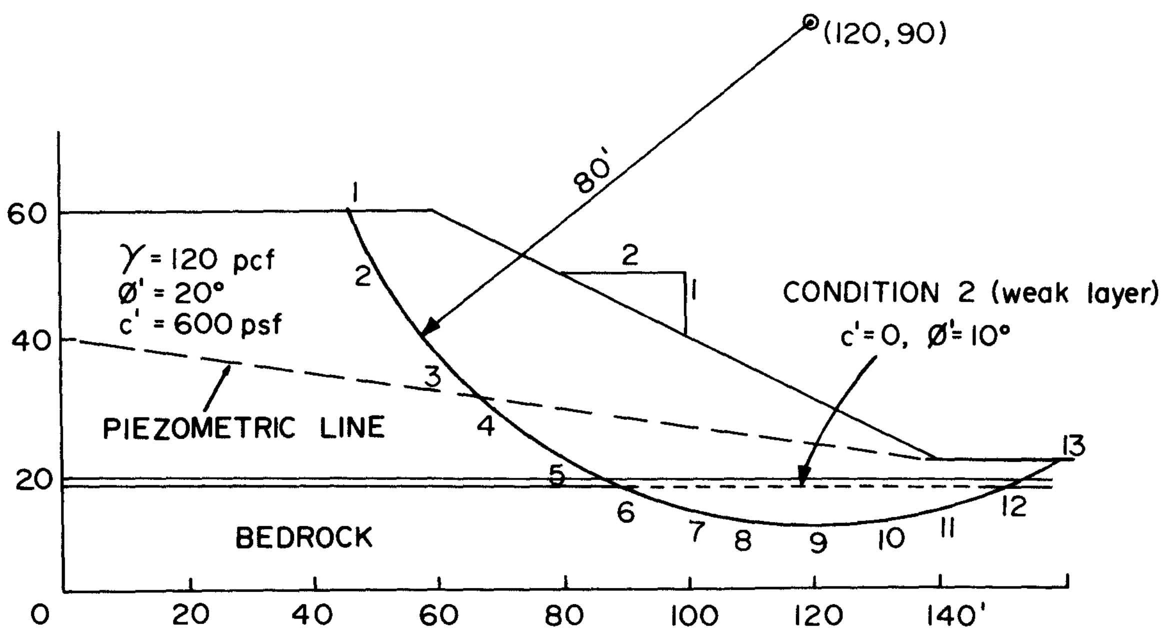


FIG. 8 EXAMPLE PROBLEM

TABLE II

Comparison of Factors of Safety for Example Problem

Case No.	Example Problem*	Ordinary Method	Simplified Bishop Method	Spencer's Method			Janbu's Simplified Method	Janbu's Rigorous Method **	Morgenstern-Price Method f(x)=Constant	
				F	θ	λ			F	λ
1	Simple 2:1 slope, 40 feet high, $ø^1 = 20$, $c^1 = 600$ PSF	1.928	2.080	2.073	14.81	0.237	2.041	2.008	2.076	0.254
2	Same as 1.) with a thin, weak layer with $ø^1 = 10^0$, $c^1 = 0$ PSI	1.288	1.377	1.373	10.49	0.185	1.448	1.432	1.378	0.159
3	Same as 1.) except with $r_u = 0.25$	1.607	1.766	1.761	14.33	0.255	1.735	1.708	1.765	0.244
4	Same as 2.) except with $r_u = 0.25$ for both materials	1.029	1.124	1.118	7.93	0.139	1.191	1.162	1.124	0.116
5	Same as 1.) except with a piezometric line	1.693	1.834	1.830	13.87	0.247	1.827	1.776	1.833	0.234
6	Same as 2.) except with a piezometric line for both materials	1.171	1.248	1.245	6.88	0.121	1.333	1.298	1.250	0.097

* Width of slice is 0.5 feet and the tolerance on the nonlinear solutions is 0.001
** The line of thrust is assumed at 0.333

lambda. The Simplified Bishop method satisfies overall moment equilibrium with lambda equal to zero. Spencer's method has lambda equal to the tangent of the angle between the horizontal and the resultant interslice force. Janbu's factors of safety can be placed along the force equilibrium line to give an indication of an equivalent lambda value. Figures 9 and 10 show comparative plots for the first two cases shown in Table II.

The results in Table II along with those from other comparative studies show that the factor of safety with respect to moment equilibrium is relatively insensitive to the interslice force assumption. Therefore, the factors of safety obtained by the Spencer and Morgenstern-Price methods are generally similar to those computed by the Simplified Bishop method. On the other hand, the factors of safety based on overall force equilibrium are far more sensitive to the side force assumption.

The relationship between the factors of safety by the various methods remains similar whether the failure surface is circular or composite. For example, the Simplified Bishop method gives factors of safety that are always very similar in magnitude to the Spencer and Morgenstern-Price method. This is due to the small influence that the side force function has on the moment equilibrium factor of safety equation. In the six example cases, the average difference in factor of safety was approximately 0.1 percent.

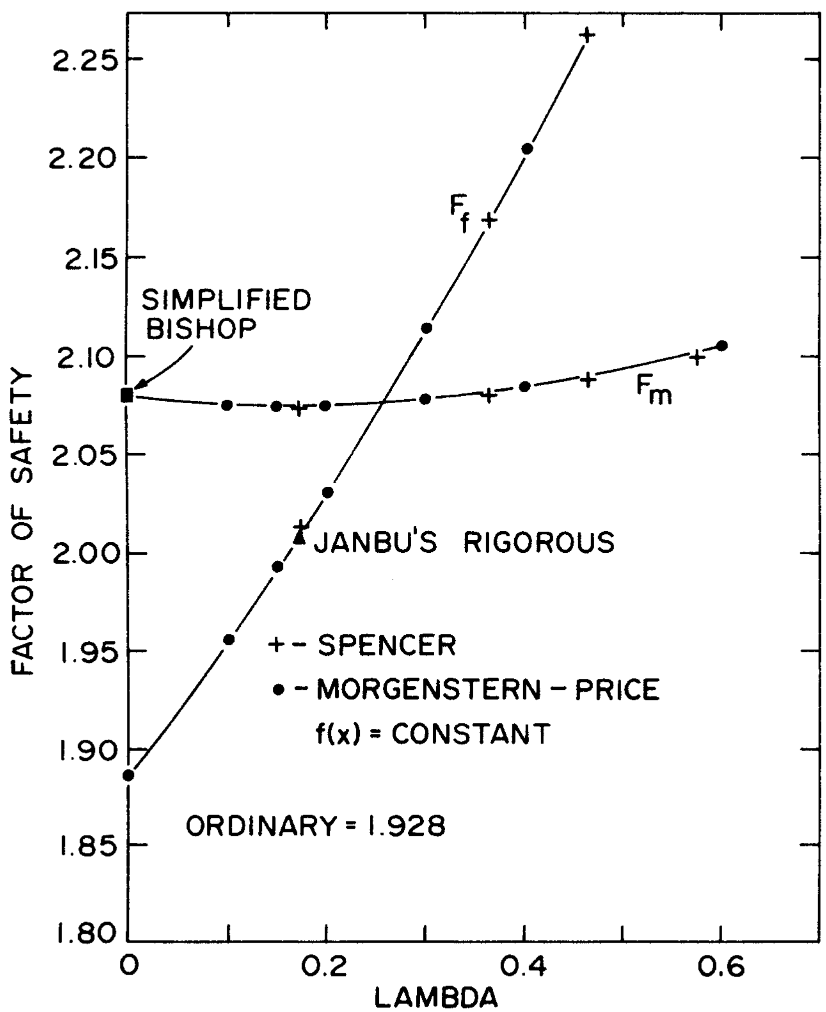


FIG. 9 COMPARISON OF FACTORS OF SAFETY FOR CASE 1

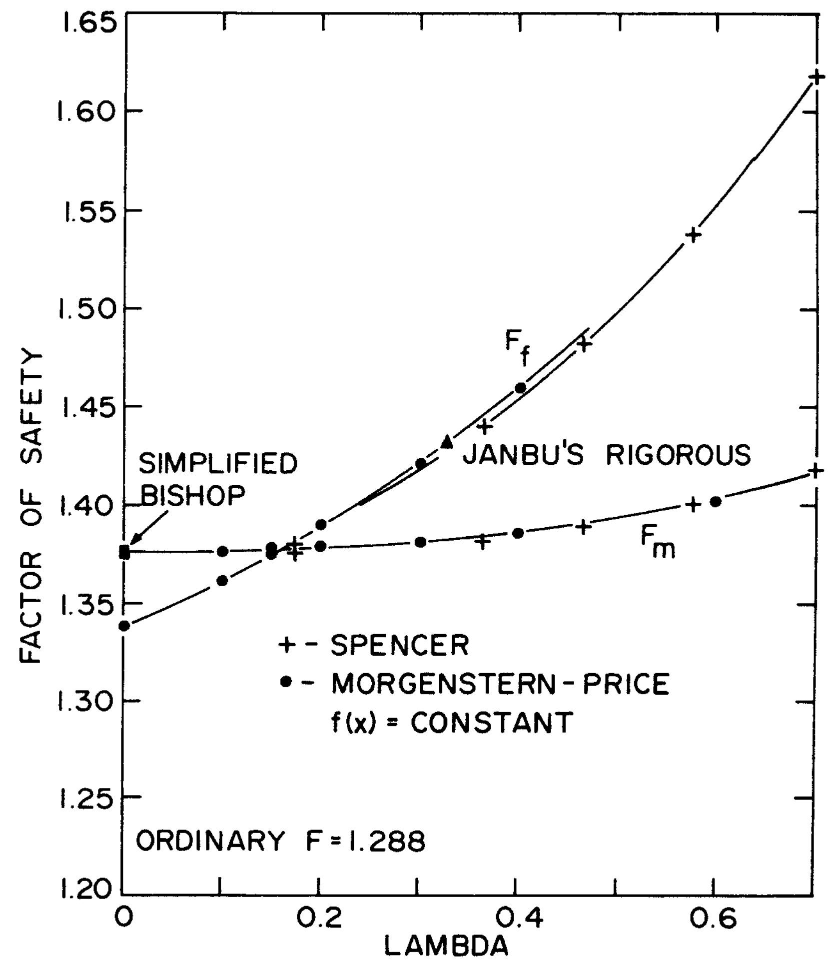


FIG. 10 COMPARISON OF FACTORS OF SAFETY FOR CASE 2

Comparison of Two Solutions to the Morgenstern-Price Method

Morgenstern-Price (1965) originally solved their method using the Newton-Raphson numerical technique. This paper has presented an alternate procedure that has been referred to as the 'Best-Fit Regression' method. The two methods of solution were compared using the University of Alberta computer program (Krahn et al, 1971) for the original method and the University of Saskatchewan computer program (Fredlund, 1974) for the alternate solution. In addition, it is possible to compare the above solutions with Spencer's method.

Table III shows a comparison of the two solutions for the example problems (Figure 8). Figures 11 and 12 graphically display the comparisons. Although the computer programs use different methods of inputting the geometry and side force function and different techniques for solving the equations, the factors of safety are essentially the same.

The example cases show that when the side force function is either a constant or a half sine, the average factor of safety from the University of Alberta computer program differs from the University of Saskatchewan computer program by less than 0.7 percent. Using the University of Saskatchewan program, the Spencer method and the Morgenstern-Price method (for a constant side force function) differ by less than 0.2 percent. The average lambda values computed by the two programs differ by approximately 9 percent. However, as shown above, this difference does not significantly affect the final factor of safety.

TABLE III

Comparison of Two Solutions to the Morgenstern-Price Method**

Case No.	Example Problem	University of Alberta Program				University of Saskatchewan SLOPE Program					
		Side Force Function				Side Force Function					
		Constant		Half Sine		Constant		Half Sine		Clipped Sine*	
		F	λ	F	λ	F	λ	F	λ	F	λ
1	Simple 2:1 slope, 40 feet high, $\phi^1 = 2\ C^o$, $C^1 = 600$ PSF	2.085	0.257	2.085	0.314	2.076	0.254	2.076	0.318	2.083	0.390
2	Same as 1.) with a thin, weak layer with $\phi^1 = 10^o$, $c^1 = 0$ PSI	1.394	0.182	1.386	0.218	1.378	0.159	1.370	0.187	1.364	0.203
3	Same as 1.) except with $r_u = 0.25$	1.772	0.351	1.770	0.432	1.765	0.244	1.764	0.304	1.779	0.417
4	Same as 2.) except with $r_u = 0.25$ for both materials	1.137	0.334	1.117	0.441	1.124	0.116	1.118	0.130	1.113	0.138
5	Same as 1.) except with a piezometric line	1.838	0.270	1.837	0.331	1.833	0.234	1.832	0.290	1.832	0.300
6	Same as 2.) except with a piezometric line for both materials	1.265	0.159	Not Converging		1.250	0.097	1.245	0.101	1.242	0.104

* Coordinates x = 0, y = 0.5 and x = 1.0, y = 0.25

** Tolerance on both Morgenstern-Price Solutions is 0.001

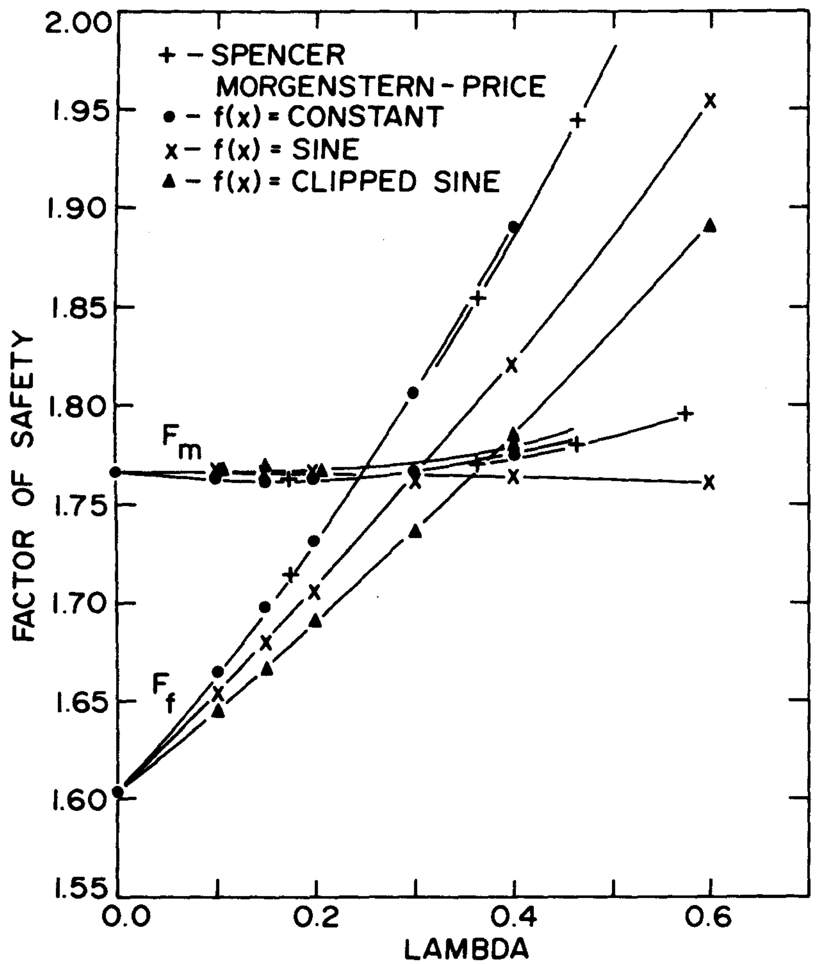


FIG. II EFFECT OF SIDE FORCE FUNCTION ON FACTOR OF SAFETY FOR CASE 3

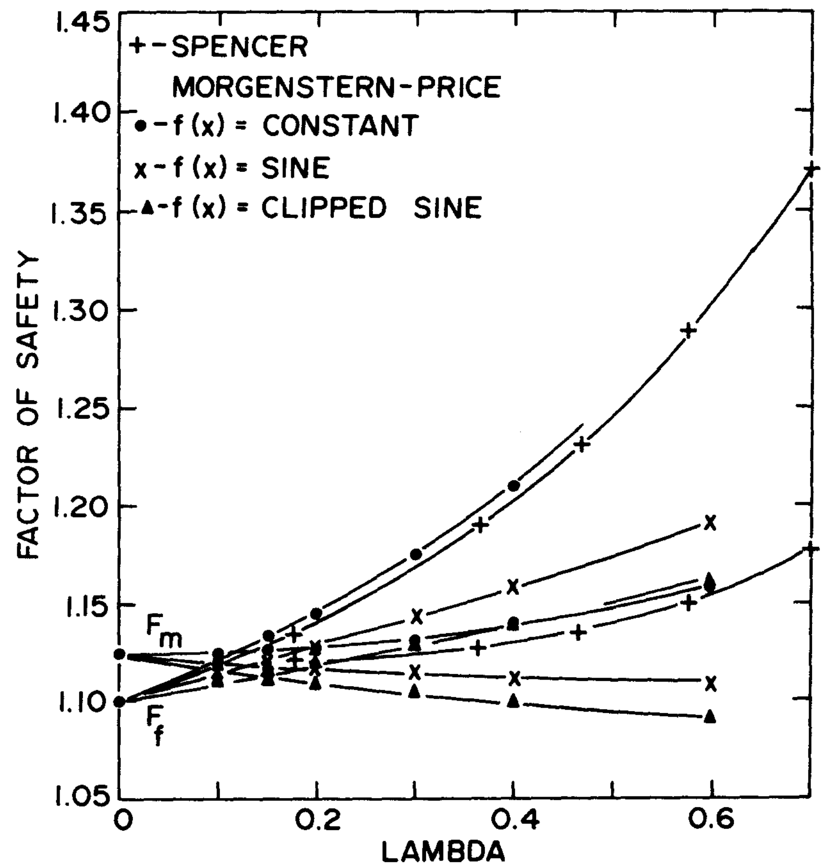


FIG. 12 EFFECT OF SIDE FORCE FUNCTION ON FACTOR OF SAFETY FOR CASE 4

Comparison of Computing Costs

A simple 2:1 slope was selected to compare the computer costs (i.e. CPU Time) associated with the various methods of analysis. The geometry was 440 feet long and divided into 5 foot slices. The results shown in Figure 13 were obtained using the University of Saskatchewan SLOPE program run on an IBM 370 model 158 computer.

The Simplified Bishop method required 0.012 minutes per stability analysis. The Ordinary method required approximately 60 percent as much time. Tha factor of safety by Spencer's method was computed using four side force angles. Each angle required 0.024 minutes. The factor of safety by the Morgenstern-Price method was computed using six lambda values. Each trial required 0.021 minutes. At least three estimates of the side force angle or lambda value are required to obtain the factor of safety. Therefore, the Spencer or Morgenstern-Price methods are at least six times as costly to run as the Simplified Bishop method. The above relative costs are slightly affected by the width of slice and the tolerance used in solving the nonlinear factor of safety equations.

Conclusions

1.) The factor of safety equations for all methods of slices considered can be written in the same form when recognizing whether moment and/or force equilibrium is explicitly satisfied. The normal force equation is of the same form for all methods with the exception of the Ordinary method. The method of handling the interslice forces differentiates the normal force equations.

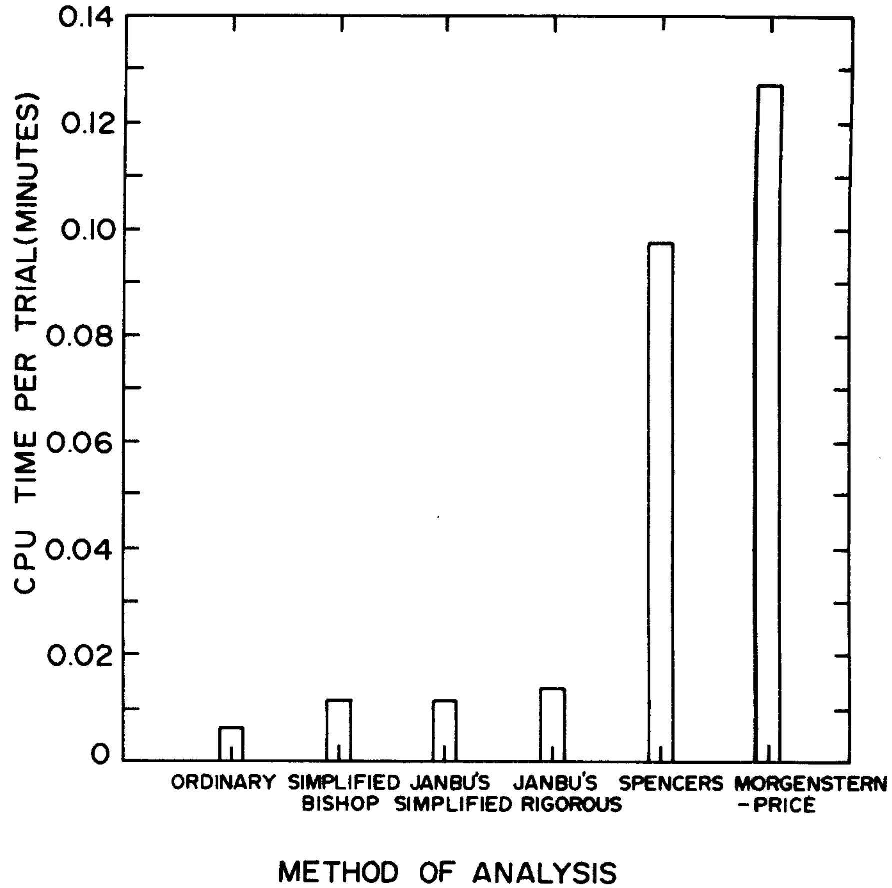


FIG. 13 TIME PER STABILITY ANALYSIS TRIAL FOR ALL METHODS OF ANALYSIS

2.) The analytical aspects of slope stability can be viewed in terms of one factor of safety equation satisfying overall moment equilibrium and another satisfying overall force equilibrium for various lambda values. Then each method becomes a special case of the 'Best-Fit' factor of safety lines.

3.) The 'Best-Fit Regression' solution and the 'Newton-Raphson' solution give the same factors of safety. They differ only in the manner in which the equations of statics are utilized.

4.) The 'Best-Fit Regression' solution is readily comprehended. It also gives a complete understanding of the variation of factor of safety with respect to lambda.

Acknowledgements

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References

BISHOP, A.W. (1955) *The Use of the Slip Circle in the Stability Analysis of Slopes,* Geotechnique, Volume 5, No. 1, pp. 7-17.

FELLENIUS, W. (1936) *Calculation of the Stability of Earth Dams,* Proceedings of the Second Congress on Large Dams, Vol. 4, pp. 445-463.

FREDLUND, D.G. (1974) *Slope Stability Analysis,* User's Manual CD-4 Department of Civil Engineering, University of Saskatchewan, Saskatoon.

FREDLUND, D.G. (1975) *A Comprehensive and Flexible Slope Stability Program,* Presented at the Roads and Transportation Association of Canada Meeting, Calgary.

JANBU, N., L. BJERRUM and B. KJAERNSLI (1956) *Stabilitetsberegning for Fyllinger Skjaeringer og Naturlige Skraninger,* Norwegian Geotechnical Publication No. 16, Oslo.

KRAHN, J., V.E. PRICE and N.R. MORGENSTERN (1971) *Slope Stability Computer Program for Morgenstern-Price Method of Analysis,* User's Manual No. 14, University of Alberta, Edmonton.

MORGENSTERN, N.R. and V.E. PRICE (1965) *The Analysis of the Stability of General Slip Surfaces,* Geotechnique, Vol. 15, pp. 70-93.

SPENCER, E. (1967) *A Method of Analysis of the Stability of Embankments Assuming Parallel Inter-Slice Forces,* Geotechnique, Vol. 17, pp. 11-26 (March).

TERZAGHI, K, and R.B. PECK (1967) *Soil Mechanics in Engineering Practice,* Second Edition. John Wiley and Sons, Inc., New York.

WHITMAN, R.V. and W.A. BAILEY, (1967) *Use of Computer for Slope Stability Analysis,* Journal of the Soil Mechanics and Foundation, Division of ASCE, Vol. 93, No. SM4 July.

WRIGHT, S. (1969) *A Study of Slope Stability and the Undrained Shear Strength of Clay Shales,* thesis presented to the University of California, at Berkely, Calif, in partial fulfillment of the requirements for the degree of Doctor of Philosophy.