Proceedings of the 2nd Symposium, Application of Solid Mechanics, McMaster University, pp. 46-59. Hamilton, Ontario. 1974

ENGINEERING APPROACH TO SOIL CONTINUA

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SUMMARY

The analysis of multiphase continua is considered from a macroscopic, phenomenological standpoint. The case of an unsaturated soil is used to illustrate the approach. The number of independent phases that comprise the system is established on the basis of independent properties of the material and the continuity of their boundaries. The fundamental laws of physics (i.e., the conservation of energy and mass) are used to derive the stress and deformation state variables required to characterize the continuum. The stress state variables must be in terms of measurable quantities and experimentally verified by means of Null type tests. Numerous tests on unsaturated soils verified the proposed stress state variables to a high degree of accuracy. The deformation state variables which are based directly on a volumetric requirement for the multiphase media, do not require experimental verification. The stress and deformation state variables can be linked by suitable constitutive relationships which are then experimentally tested for uniqueness. For the case of isotropic loading of an unsaturated soil, two constitutive relationships are required. Laboratory results showed good agreement between predicted and measured deformations of the soil structure for two soils tested. The agreement for the water phase constitutive surface was not quite as good. Thus, the multiphase media can be analyzed within the fundamental context of continuum mechanics to form a sound basis for solving practical engireering problems.

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INTRODUCTION

Many materials in nature exist as mixtures of several phases. Of particular importance are soil multiphase continua such as:

i) saturated soils ii) unsaturated soils

iii) partly frozen soils iv) oil sands

v) fractured rock masses and others.

The analysis of a particulate system could be considered on an interparticle basis; however, then particle physics or statistical mechanics must be employed. The limited degree of success in their application in engineering discourages this consideration [1]. Intuitively, it appears advantageous to investigate the behavior of particulate systems from a macroscopic, phenomenological standpoint.

The basic engineering problem related to the above materials involves the prediction of motion of each phase of the continuum. Forces applied to solids cause deformation, and forces applied to liquids cause flow; however, in each case, the engineer is interested in mapping movements.

The engineering behavior of multiphase continua is herein considered within the context of multiphase continuum mechanics. The proposed analysis involves four steps applicable to all materials. These are:

i) the description of the physical element. Special consideration is given to inter-phases between the bulk phases.

ii) the establishment of the state variables associated with each phase. These can be theoretically predicted on the basis of the fundamental laws of continuum mechanics and experimentally verified.

iii) the proposal of suitable constitutive relationships and the experimental verification of their uniqueness.

iv) the formulation of solutions to engineering problems and the comparison of the theoretical results with actual case histories.

The above procedure for analyzing multiphase systems is particularly pertinent to the field of soil mechanics. The case of an unsaturated soil is used as an example in considering the above procedure.

ELEMENT DESCRIPTION AND TERMINOLOGY

The element of the continua under consideration should be completely enclosed and of a size that can be disassembled into all its independent phases [2]. The fundamental laws of continuum mechanics can then be applied to each phase [1],[3]. In the past, part of the problem in applying this procedure has resulted from a failure to recognize all the phases involved. For example, the inter-phases between the bulk phases have not been treated as independent phases even though their effects on soil behavior have been recognized.

Two factors must be considered when delineating the phases [4],[5]. The inter-phases should be considered as independent phases when they:

i) possess properties differing from the contiguous homogeneous phases

ii) have continuous bounding surfaces throughout the element.

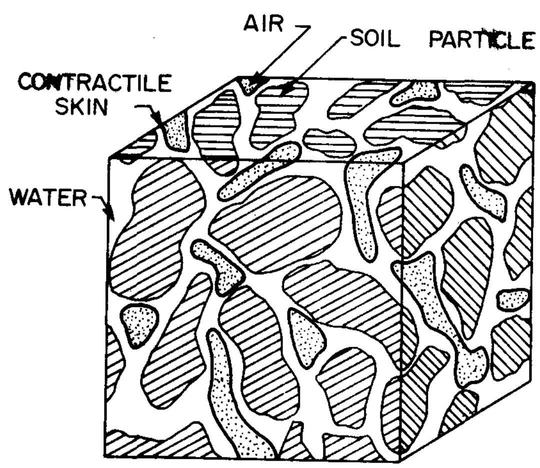

FIGURE 1 ELEMENT OF AN UNSATURATED SOIL

On the basis of the above factors, let us postulate that the air-water inter-phase (commonly referred to as the "contractile skin") be considered as an independent phase [6]. An element of unsaturated soil can, therefore, be visualized as a mixture with two phases that come to equilibrium under applied stresses (i.e., soil particles and the contractile skin) and two phases that flow under applied pressures (i.e., the air and water). Figure 1 depicts the element under consideration.

STRESS STATE VARIABLES

The success of the effective stress equation in describing the behavior of saturated soils has led research workers into a search for a similar statement for unsaturated soils. During the last two decades there have been numerous equations proposed in the literature; however, none have proven satisfactory in practice. The only attempt at a theoretical justification for any of the proposed equations has involved statical equilibrium across a wavey plane passed through the soil. The resulting analysis merely produces an equivalence statement with no fundamental justification.

The mechanical processes associated with engineering problems are governed by:

i) the conservation of mass; and

ii) the equation of motion.

The variables associated with these fundamental equations are defined as "state variables". They are required for the fundamental characterization of the continuum and are independent of the type of material involved.

The procedure advocated herein for the prediction of "stress state variables" is based on the equations of motion, (or indirectly the conservation of energy) and utilizes the principle of the superposition of coincident equilibrium stress fields from which "stress state variables" can be extracted [7],[8],[9]. It is assumed that each phase has an independent continuous stress field associated with it. Therefore, the number of independent equations is equal to the cartesian coordinate directions multiplied by the number of phases in the continuum. An additional overall or total stress field can be written for the assembled element since the equilibrium equations associated with each phase are linear.

Although equilibrium equations can be readily written for any number of phases, the extracted "stress state variables" will be meaningful only if they are measurable quantities. In the case of a saturated soil, the particle stress field cannot be directly measured. However, the equilibrium equation for the particle phase can be written in terms of the difference between the total stress field and the water phase stress field. A similar procedure has been used for an unsaturated soil by Fredlund [10].

Of main interest in an unsaturated soil case, is the equilibrium equation associated with the soil particles and the contractile skin. The analysis shows that the two equilibrium equations are the same. In the y-direction,

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial(\sigma_y - u_w)}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} + (\gamma_t - \gamma_w) + \frac{F_{cwy} + F_{pwy}}{n_w}$$

$$+ n_a \cdot \frac{\partial(u_a - u_w)}{\partial y} + F_{cay} + n_a \cdot \gamma_a = 0 \qquad (1)$$

where τ_{xy}, τ_{zy} = total shear stresses in the y-direction

σ_y = total stress in the y-direction

u_w = water pressure

u_a = air pressure

γ_t, γ_w, γ_a = density of the total mixture, the water and the air phases, respectively

n_w, n_a = porosities or percentage of the element composed of water and air, respectively

F_{cwy} = interaction body force of the contractile skin on the water phase in the y-direction.

F_{pwy} = interaction body force of the particles on the water phase in the y-direction.

F_{cay} = interaction body force of the contractile skin on the air phase in the y-direction.

Similar equilibrium equations can be written for the x and z-directions and the extracted "stress state variables" form two independent stress matrices.

$$\begin{bmatrix} \sigma_x - u_w & \tau_{yx} & \tau_{zx} \\ \tau_{xy} & \sigma_y - u_w & \tau_{zy} \\ \tau_{xz} & \tau_{yz} & \sigma_z - u_w \end{bmatrix}$$

and

$$\begin{bmatrix} u_a - u_w & 0 & 0 \\ 0 & u_a - u_w & 0 \\ 0 & 0 & u_a - u_w \end{bmatrix}$$

The theoretical analysis further reveals that any two of a possible three normal "stress state variables" (i.e., $\sigma - u_w$, $\sigma - u_a$, and $u_a - u_w$) can be used to describe the stress state of the soil particles and the contractile skin of an unsaturated soil.

The theoretically proposed "stress state variables" can be experimentally verified by means of Null type tests [11]. If the above stress tensors are correct, equilibrium in the soil structure and contractile skin should be maintained as long as each "stress state variable" remains constant. This means that if the normal stress components of the stress state variables could be changed in such a manner that the stress state variables remain constant, equilibrium should be maintained. The Null test can be stated:

$$\Delta\sigma_x = \Delta\sigma_y = \Delta\sigma_z = \Delta u_w = \Delta u_a$$

If volume changes associated with the soil structure and the contractile skin remain at zero in an open system, equilibrium has been maintained and the theoretically proposed stress tensors are experimentally verified.

The experimental data from a large number of tests indicated essentially no volume change during the Null tests [11]. The final set of tests indicated that the volume changes were less than 1 part in 10,000 parts, even after several days duration (Figures 2 and 3). Thus, the proposed "stress state variables" are well verified.

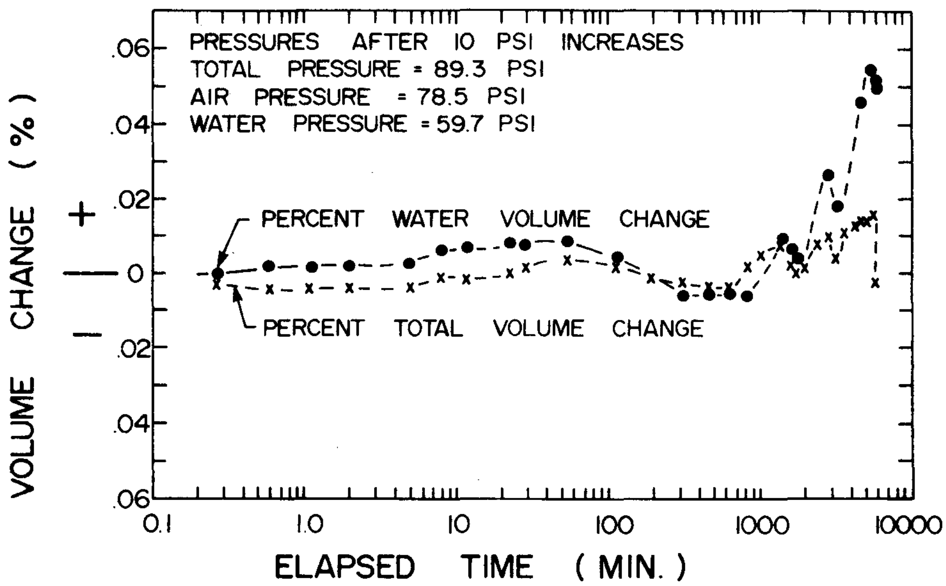

FIGURE 2 NULL TEST (N-37) ON UNSATURATED SAMPLE NO. 31

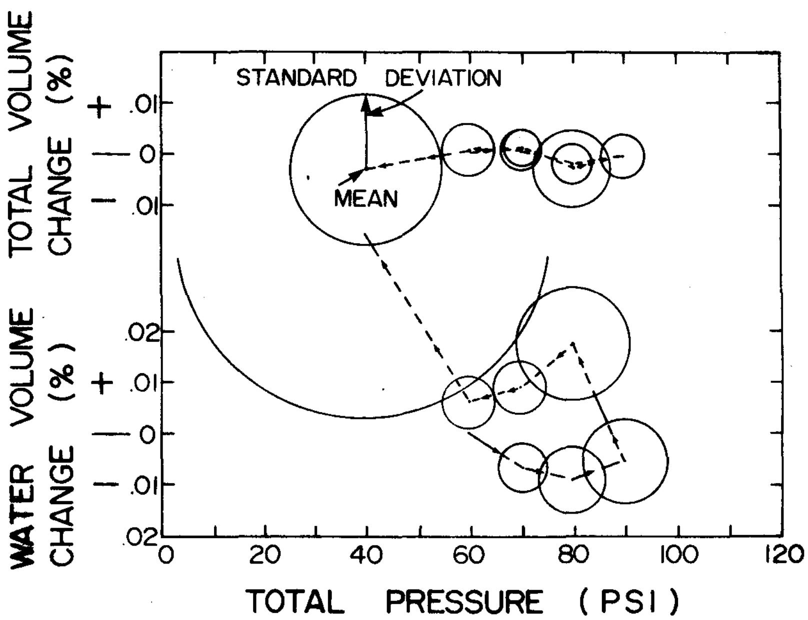

FIGURE 3 MEANS AND STANDARD DEVIATIONS WITH RESPECT TO TIME FOR THE TOTAL AND WATER VOLUME CHANGES FROM ALL NULL TESTS ON SAMPLE NO. 31

DEFORMATION STATE VARIABLES

The deformation state variables required for mapping motion in each of the phases are formulated on the basis of the conservation of mass (i.e., $dm/dt = 0$). Since at any time, the mass of a multiphase continuum is equal to the sum of the masses of the components, the average density of the mixture can be expressed as the sum of the masses of the various phases divided by the sum of the volumes of the phases [1]. This gives rise to the volumetric requirement that at any time,

$$V = \Sigma V_i \tag{2}$$

where V = total volume

V_i = volume of each phase at any time.

When considering changes in volumes, either a referential or spatial type of element can be used. For multiphase continua, either a consistent spatial element can be used for all phases [12] or a referential element can be used for one of the phases. The volumetric requirement for either element is,

$$\frac{\Delta V}{V} = \frac{\Sigma \Delta V_i}{V} \tag{3}$$

In the case of an unsaturated soil, the volumetric requirement is,

$$\frac{\Delta V}{V} = \frac{\Delta V_p}{V} + \frac{\Delta V_w}{V} + \frac{\Delta V_a}{V} + \frac{\Delta V_c}{V} \tag{4}$$

where V = total volume of the element

V_p = volume of soil particles

V_w = volume of water

V_a = volume of air

V_c = volume of contractile skin

If we assume that changes in the volume of the soil particles are small, and that changes in the volume of the contractile skin are internal to the element, the volumetric requirement reduces to,

$$\frac{\Delta V}{V} = \frac{\Delta V_w}{V} + \frac{\Delta V_a}{V} \tag{5}$$

The volumetric requirement shows that the volume changes of any two of the above three variables must be measured while the third can be computed. Suitable deformation state variables can now be defined within the context of the volumetric requirement. The water and air phases require only one volumetric deformation state variable. For example, let,

$$\theta_w = \frac{\Delta V_w}{V} \text{ and } \theta_a = \frac{\Delta V_a}{V} \tag{6}$$

For the overall element (i.e., soil structure), a suitable deformation matrix is,

$$\begin{bmatrix} \varepsilon_x & \varepsilon_{yx} & \varepsilon_{zx} \\ \varepsilon_{xy} & \varepsilon_y & \varepsilon_{zy} \\ \varepsilon_{xz} & \varepsilon_{yz} & \varepsilon_z \end{bmatrix}$$

where ε_x, ε_y, ε_z = normal deformation state variables

ε_{xy}, ε_{xz}, ε_{yz} and ε_{yx}, ε_{zx}, ε_{zy} = shear deformation state variables.

It does not appear necessary to experimentally verify the deformation state variables.

When the "time" variable is incorporated into the analysis, the conventional continuum mechanics continuity equation applies to each phase (i.e., there are "n" independent continuity equations). If it is applied to each phase, it will also satisfy the above volumetric requirement.

CONSTITUTIVE RELATIONSHIPS

Suitable constitutive relationships can now be proposed which link the stress and deformation state variables. They can be proposed either strictly from a mathematical standpoint (i.e., linear combinations of the state variables) or from a semi-empirical standpoint which is based on observed behavior [13]. The proposed constitutive equations must then be experimentally verified for uniqueness.

For the case of an isotropically loaded, unsaturated soil, two constitutive equations are necessary to describe volume changes (Figure 4). For the soil structure, let us propose a general, incremental constitutive relationship that takes the form,

$$\varepsilon = \frac{1}{V}\cdot\frac{\partial V}{\partial(\sigma - u_w)}\cdot d(\sigma - u_w) + \frac{1}{V}\cdot\frac{\partial V}{\partial(u_a - u_w)}\cdot d(u_a - u_w) \qquad (7)$$

where ε = volumetric strain of the overall element or the soil structure

$\frac{1}{V}\cdot\frac{\partial V}{\partial(\sigma - u_w)}$ = compressibility of the soil structure with respect to changes in $(\sigma - u_w)$

$\frac{1}{V}\cdot\frac{\partial V}{\partial(u_a - u_w)}$ = compressibility of the soil structure with respect to changes in $(u_a - u_w)$

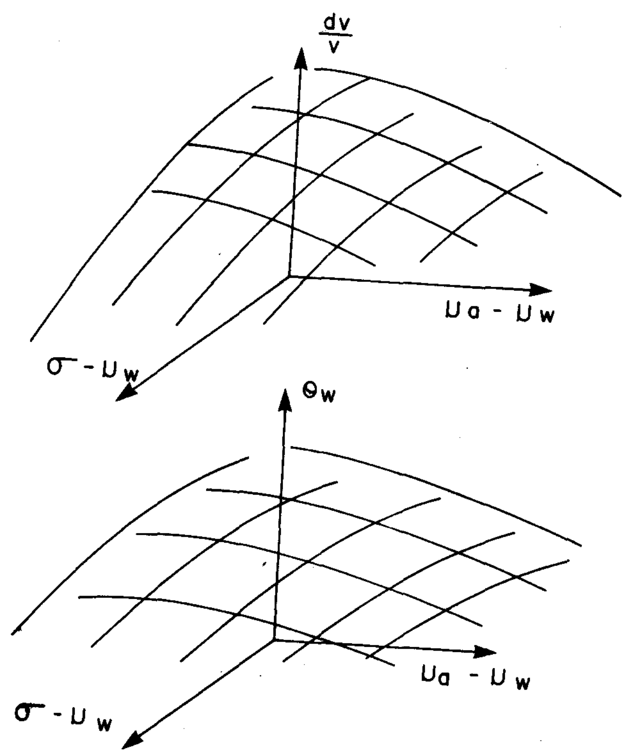

FIGURE 4 CONSTITUTIVE SURFACES FOR THE SOIL STRUCTURE AND THE WATER PHASE (ISOTROPIC LOADING)

Let us write the second incremental constitutive relationship for the water phase,

$$\theta_w = \frac{1}{V}\cdot\frac{\partial V_w}{\partial(\sigma - u_w)}\cdot d(\sigma - u_w) + \frac{1}{V}\cdot\frac{\partial V_w}{\partial(u_a - u_w)}\cdot d(u_a - u_w) \qquad (8)$$

where $\frac{1}{V}\cdot\frac{\partial V_w}{\partial(\sigma - u_w)}$ = slope of the water volume versus $(\sigma - u_w)$ plot

$\frac{1}{V}\cdot\frac{\partial V_w}{\partial(u_a - u_w)}$ = slope of the water volume versus $(u_a - u_w)$ plot

The uniqueness of the proposed constitutive surfaces should be analyzed in two steps. First, the uniqueness of each constitutive surface should be tested for small deviations from a stress state point. Secondly, each constitutive surface should be explored in terms of larger stress state variable changes and reversals of stress.

Let us consider only the first test by assuming that we have several identical, unsaturated soil samples subjected to the same stress conditions. Stress state variable changes in any direction should describe a planar surface (Figure 5). Thus, a knowledge of any two slopes on the constitutive surface can be used to predict movements in any other direction. Figure 6 shows the correlation between the predicted and measured deformations of the soil structure on Regina clay. The close agreement demonstrates the uniqueness of the constitutive surface. Similar checks should be made for other soil types. The uniqueness of the water phase constitutive surface for Regina clay was not as close. However, the uniqueness appears adequate for engineering purposes. A more detailed analysis of the reliability of the proposed forms of the constitutive equations for Regina clay and compacted kaolin is presented by Fredlund [11].

SOLVING PRACTICAL PROBLEMS

1. Deformation Problems

The classical elasticity approach to solving deformation problems involves the determination of the stresses, strains and displacements as functions of the space co-ordinates. The available equations to assist in formulation are the equilibrium equations, the deformation-displacement relations and the constitutive relations. Various combinations of the governing equations can be used to solve deformation problems [13]. The common "displacement" and "stress" formulations result in satisfactory field equations. In addition, the derived field equations must satisfy prescribed stress or displacement boundary conditions. The field equations can be solved as a closed form equation for simple, one-dimensional problems or by a numerical technique for more complex problems.

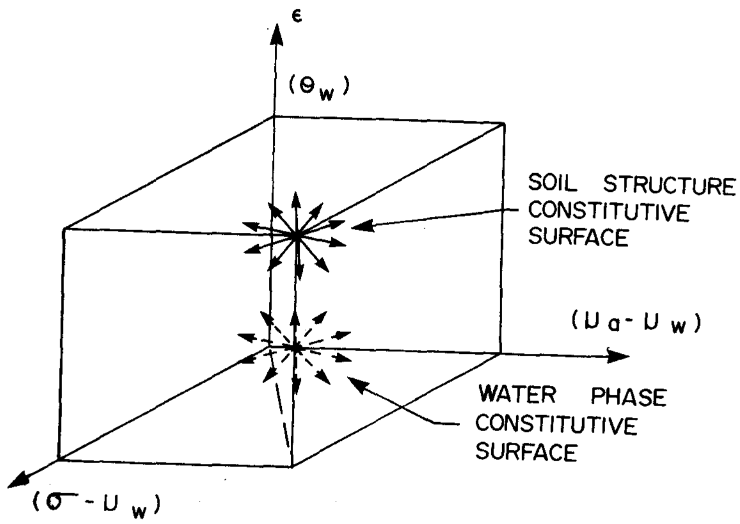

FIGURE 5 IDEAL TEST PROCEDURE TO PROVE UNIQUENESS OF THE CONSTITUTIVE SURFACE AT A POINT FOR THE SOIL STRUCTURE AND THE WATER PHASE

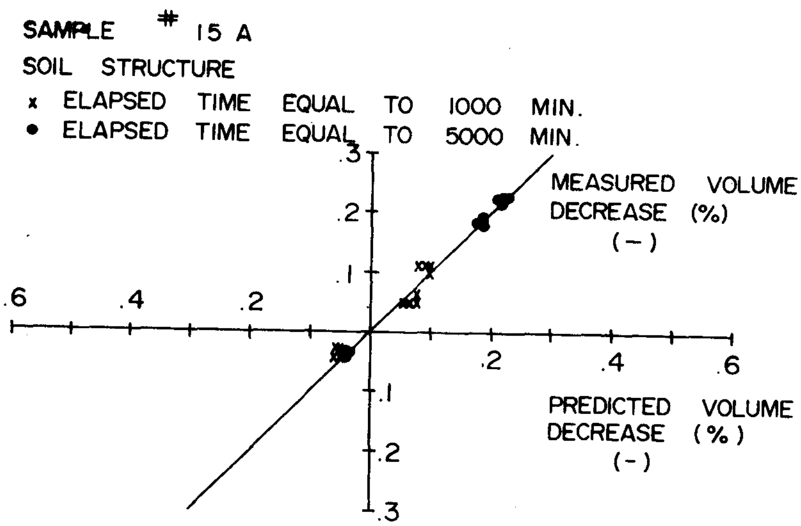

FIGURE 6 COMPARISON OF THE PREDICTED AND MEASURED SOIL STRUCTURE DEFORMATIONS FOR SAMPLE #15A (REGINA CLAY)

To solve practical problems, the engineer must be able to:

i) evaluate or predict the initial and final stress or deformation state variables; and

ii) evaluate the soil parameters either by insitu or laboratory techniques.

2. Transient Problems

In transient problems, the state variables are a function of time. These types of problems can be considered by substituting the constitutive (i.e., stress-deformation equations) and suitable flow law equations into the continuity requirements for the multiphase continua.

3. Strength Problems

A stress state variable approach can also be used to consider shear strength problems. For example, the strength of an unsaturated soil would be described in terms of the stress state variables. After evaluating the shear strength soil parameters, the strength related to any set of stress conditions can be predicted. (The proposed strength surface would need to be tested for uniqueness).

For any of the above problems, it is necessary to compare results from the theoretical formulations with actual case histories. It appears that in the past, many of the problems arising in the analysis of multiphase continua are the result of an improper description and understanding of the phases of the continua. This paper has concentrated on the basic approach that must be taken towards multiphase continua in order that the formulations are consistent with the fundamental laws of continuum mechanics.

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