Soil-Water Characteristic Curve Equation with Independent Properties

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Abstract: The soil-water characteristic curve (SWCC) has traditionally been represented using equations whose fitting parameters do not individually correspond to clearly defined soil properties or to features of the curve. As a result, unique sets of parameters are often nonexistent, and sensitivity analyses and statistical assessments of SWCC parameters become difficult. In order to overcome these difficulties, a new class of equations to represent unimodal and bimodal SWCCs is proposed. The chosen fitting parameters are the air-entry value, the residual suction, the residual degree of saturation, and a parameter that controls the sharpness of the curvatures. The physical meaning for the soil parameters is discussed for different soil types. A unique relation between each of the equation parameters and the individual features of SWCCs is assured. The proposed equations are fitted to data corresponding to a variety of soil types and a good fit is observed.

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Introduction

Appropriate equations to mathematically represent soil-water characteristic curves (SWCCs) are required for both graphical presentations and for numerical modeling. Leong and Rahardjo (1997) and Sillers et al. (2001) have presented reviews of a range of proposed unimodal equations along with parametric analyses. Difficulties in the application of the available equations exist because the parameters of these equations are not individually related to shape features of the SWCC. As a result, unique values of the soil parameters are difficult to determine and sensitivity analyses become awkward. Statistical assessments of SWCCs and the grouping of soils with typical fitting parameters also become difficult (Fredlund et al. 2000). The lack of physical meaning for the fitting parameters is also undesirable.

This technical note proposes a new class of equations based on parameters that are independently related to well-defined features of the shape of typical SWCCs. The selected features possess clear physical meaning. The new equations have been developed for both unimodal and bimodal curves. The mathematical basis for the equations is described and the properties and capabilities of the equations are demonstrated.

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Appropriate Soil-Water Characteristic Curve Equation Parameters

This section describes SWCC features that can be used as equation parameters and their physical meaning. SWCCs are presented in terms of degree of saturation, S, plotted on arithmetic scale. Soil suction, ψ, is plotted on a log scale from 0.1 to 1,000,000 kPa; the latter corresponding to the completely dry conditions. The following comments consider only desaturation, but the parameters chosen for the proposed equation are valid for both wetting and drying curves.

Unimodal Curves

Soils with different textures and/or pore-size distributions have different SWCCs, as illustrated in Fig. 1. Sandy soils, represented by curve 1a, remain essentially saturated up to the so-called, air-entry value, ψ_b, where the largest pores start draining (Brooks and Corey 1964; White et al. 1970). From this point, the steeper the slope, the narrower the pore-size distribution. Once the second bending point (given by the residual degree of saturation, S_{res}, and residual soil suction, ψ_{res}) is reached, large increments in suction have relatively little effect on S. Silty soils, represented in Fig. 1 by curve 1b, have SWCCs similar to those of sandy soils, but ψ_b and ψ_{res} are usually higher due to the presence of smaller pores.

Clayey soils (curves 1c and 1d in Fig. 1) have air-entry values higher than those of silty and sandy soils and residual points that cannot always be visually identified. Adsorptive forces influence the SWCC for almost the entire range of soil suction (Mitchell 1976) and vapor flow has an important role on the moisture transfer past the residual point (Barbour 1998). Therefore the capillary theory cannot fully explain the SWCC behavior of clayey soils.

Regardless of the physical meaning of the chosen soil parameters, ψ_b, ψ_{res}, and S_{res} are distinct features of the shape of typical unimodal SWCCs and are therefore appropriate, well-defined

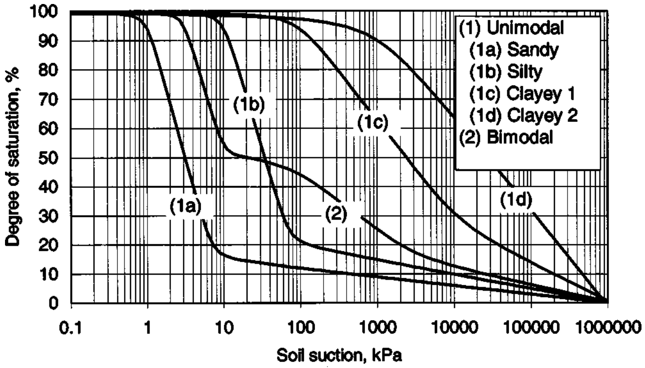

Fig. 1. Soil-water characteristic curves conceptualizations for various soil textures

soil parameters. A fourth parameter, here called "a," defines the sharpness of the transitions at both bending points. The type of SWCCs represented in Fig. 1 by curve 1d requires the parameters ψ_b and a.

Bimodal Curves

Curve 2 in Fig. 1 illustrates a typical bimodal SWCC. According to the capillary theory, the double "hump" can be associated with a bimodal pore-size distribution. Bimodal pore-size distributions are often related to gap-graded grain-size distributions (Durner 1994), but they are also observed in certain structured soils (Camapum de Carvalho et al. 2002).

Two distinct air-entry values and two distinct residual points can be defined for bimodal SWCCs, giving a total of four bending points. An additional parameter, a, is again used to define the sharpness of the transitions at the bending points. In summary, eight parameters are identified to represent bimodal curves: ψ_{b1}, $\psi_{\text{res}1}$, $S_{\text{res}1}$, ψ_{b2}, S_b, $\psi_{\text{res}2}$, $S_{\text{res}2}$, and a.

Proposed Soil-Water Characteristic Curve Equations

Three equations are proposed; namely, (1) unimodal equation with one bending point; (2) unimodal equation with two bending points; and (3) bimodal equation. The equations are based on the general hyperbole equation in the coordinate system log(ψ)-S. The equation parameters are defined as the coordinates where the hyperbolas asymptotes meet. Therefore a meaningful and consistent geometrical relationship exists between the shape of the SWCC and the equation parameters.

Wetting curves that achieve a maximum degree of saturation of less than $S=100\%$ can be represented by multiplying the proposed equations by the maximum degree of saturation. SWCCs represented in terms of gravimetric or volumetric water content can be modeled in a similar way.

Unimodal Equation with One Bending Point

One rotated and translated hyperbole is used to represent this first type of SWCC curve. The two straight lines defined by (0, 1), $(\psi_b, 1)$, and $(10^6, 0)$ are the hyperbole asymptotes. The equation is written as follows:

$$S=\frac{\tan\theta(1+r^2)\ln(\psi/\psi_b)}{(1-r^2\tan^2\theta)} - \frac{(1+\tan^2\theta)}{(1-r^2\tan^2\theta)}\sqrt{r^2\ln^2(\psi/\psi_b)+\frac{a^2(1-r^2\tan^2\theta)}{(1+\tan^2\theta)}}+1 \tag{1}$$

where $\theta=-\lambda/2$=hyperbole rotation angle; $r=\tan(\lambda/2)$=aperture angle tangent; and $\lambda=\arctan\{1/[\ln(10^6/\psi_b)]\}$=desaturation slope.

The first derivative of Eq. (1) with respect to ψ is required to define "water storage" in transient seepage analyses and can be written as follows:

$$\frac{dS}{d\psi}=\frac{1}{\psi}\left[\frac{\tan\theta(1+r^2)}{1-r^2\tan^2\theta} - \frac{r^2\ln(\psi/\psi_b)(1+\tan^2\theta)/(1-r^2\tan^2\theta)}{\sqrt{r^2\ln^2(\psi/\psi_b)+a^2(1-r^2\tan^2\theta)/(1+\tan^2\theta)}}\right] \tag{2}$$

Unimodal Equation with Two Bending Points

Two rotated and translated hyperbolas are needed to define an entire unimodal SWCC with two bending points. The three straight lines defined by (0, 1), $(\psi_b, 1)$, $(\psi_{\text{res}}, S_{\text{res}})$, and $(10^6, 0)$ are the asymptotes of the hyperbolas. These two hyperbolas are merged through a third equation, producing a continuous equation with a smooth transition. The proposed equation is as follows:

$$S=\frac{S_1-S_2}{1+(\psi/\sqrt{\psi_b\psi_{\text{res}}})^d}+S_2 \tag{3}$$

where

$$S_i=\frac{\tan\theta_i(1+r_i^2)\ln(\psi/\psi_i^a)}{(1-r_i^2\tan^2\theta_i)}+(-1)^i \times\frac{(1+\tan^2\theta_i)}{(1-r_i^2\tan^2\theta_i)}\sqrt{r_i^2\ln^2(\psi/\psi_i^a)+\frac{a^2(1-r_i^2\tan^2\theta_i)}{(1+\tan^2\theta_i)}}+S_i^a$$

$i=1,2$; $\theta_i=-(\lambda_{i-1}+\lambda_i)/2$=hyperbolas rotation angles; $r_i=\tan[(\lambda_{i-1}-\lambda_i)/2]$=aperture angles tangents; $\lambda_0=0$ and $\lambda_i=\arctan\{(S_i^a-S_{i+1}^a)/[\ln(\psi_{i+1}^a/\psi_i^a)]\}$=desaturation slopes; $S_1^a=1$; $S_2^a=S_{\text{res}}$; $S_3^a=0$; $\psi_1^a=\psi_b$; $\psi_2^a=\psi_{\text{res}}$; $\psi_3^a=10^6$; and $d=2\exp[1/\ln(\psi_{\text{res}}/\psi_b)]$=weight factor for S_1 and S_2 that produces a continuous and smooth curve.

The first derivative of Eq. (3) with respect to ψ is

$$\frac{dS}{d\psi}=\frac{dS_1/d\psi-dS_2/d\psi}{1+(\psi/\sqrt{\psi_b\psi_{\text{res}}})^d}-\frac{S_1-S_2}{[1+(\psi/\sqrt{\psi_b\psi_{\text{res}}})^d]^2} \times\left(\frac{\psi}{\sqrt{\psi_b\psi_{\text{res}}}}\right)^d\frac{d}{\psi}+\frac{dS_2}{d\psi} \tag{4}$$

where

$$\frac{dS_i}{d\psi}=\frac{1}{\psi}\left[\frac{\tan\theta_i(1+r_i^2)}{1-r_i^2\tan^2\theta_i}+(-1)^i \times\frac{r_i^2\ln(\psi/\psi_i^a)(1+\tan^2\theta_i)/(1-r_i^2\tan^2\theta_i)}{\sqrt{r_i^2\ln^2(\psi/\psi_i^a)+a^2(1-r_i^2\tan^2\theta_i)/(1+\tan^2\theta_i)}}\right]$$

$i=1, 2.$

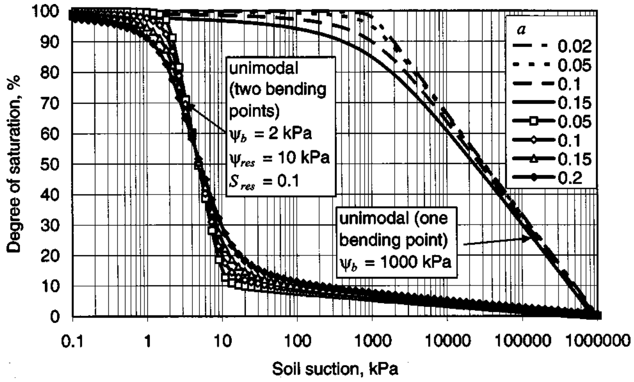

Fig. 2. Effect of changing a on unimodal curves

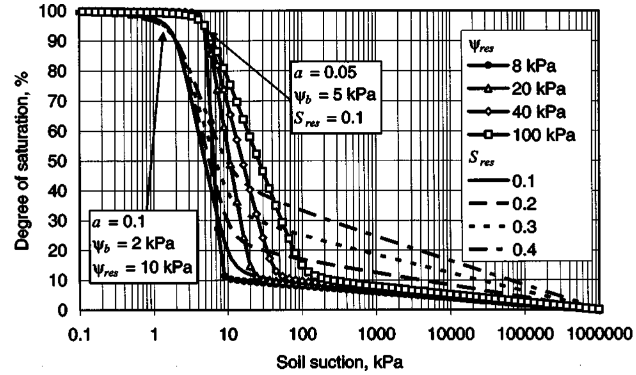

Fig. 4. Effect of changing ψ_{res} and S_{res} on the unimodal curve with two bending points

Bimodal Equation

Four hyperbolas are needed to model a bimodal SWCC delineated by the five asymptotes that are defined by (0, 1), $(\psi_{b1}, 1)$, $(\psi_{\mathrm{res}1}, S_{\mathrm{res}1})$, (ψ_{b2}, S_b), $(\psi_{\mathrm{res}2}, S_{\mathrm{res}2})$, and $(10^6, 0)$

$$S=\frac{S_1-S_2}{1+(\psi/\sqrt{\psi_{b1}\psi_{\mathrm{res}1}})^{d_1}}+\frac{S_2-S_3}{1+(\psi/\sqrt{\psi_{\mathrm{res}1}\psi_{b2}})^{d_2}}+\frac{S_3-S_4}{1+(\psi/\sqrt{\psi_{b2}\psi_{\mathrm{res}2}})^{d_3}}+S_4 \tag{5}$$

where S_i, θ_i, r_i, and λ_i were defined in Eq. (3); $i=1, 2, 3, 4$; $S_1^a=1$; $S_2^a=S_{\mathrm{res}1}$; $S_3^a=S_b$; $S_4^a=S_{\mathrm{res}2}$; $S_5^a=0$; $\psi_1^a=\psi_{b1}$; $\psi_2^a=\psi_{\mathrm{res}1}$; $\psi_3^a=\psi_{b2}$; $\psi_4^a=\psi_{\mathrm{res}2}$; $\psi_5^a=10^6$; $d_j=2\exp[1/\ln(\psi_{j+1}^a/\psi_j^a)]$ = weight factors, $j=1, 2, 3$.

The derivative of Eq. (5) with respect to ψ can be obtained in a manner similar to that above for Eq. (3).

Parametric Analysis of the Proposed Equations

Parametric studies were used to describe the fitting properties of the proposed equations. Figs. 2–4 show that when changing one parameter while keeping the others fixed, only that feature of the curve related to the parameter being varied is affected. Thus the curve parameters are mathematically independent. In this way, the new proposed equation is unique amongst all other proposed continuous SWCC equations.

As Fig. 2 illustrates, the larger the value of a the smoother the curve. As a is increased, the air-entry value might appear to be reduced, but that is not the case. Rather, the apparent reduction should be viewed as a smoothing effect evenly distributed to suction values lower and higher than both ψ_b and ψ_{res}. Ultimately, the bending points are fixed curve parameters that are totally independent on the value of a. Limits need to be imposed for the value of parameter a. When values of a greater than 0.2 are used, the curve limits may start deviating excessively from $S=100\%$ and $S=0\%$, respectively (see Fig. 2). For this reason, a range of values of a from 0 to 0.15 is suggested.

Due to its similar mathematical nature, parametric analyses of the unimodal equations suffice to demonstrate the independence of the parameters of the bimodal equation.

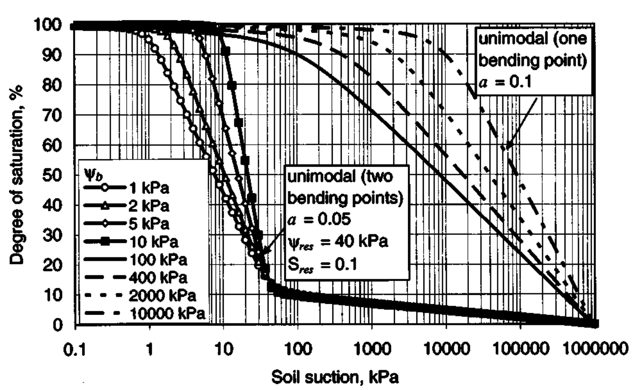

Fig. 3. Effect of changing ψ_b on unimodal curves

Fitting the Proposed Equations to Experimental Data

Experimental data sets were selected to demonstrate the fitting capabilities of the proposed equations. Since the equation parameters have clear and distinct roles, an eye fitting would be appropriate. However, in order to avoid human bias, a rigorous minimum squares fitting analysis was performed using the minimization solver available in MS *Excel 97*.

Fig. 5 presents the best-fit curves to the experimental data obtained for Regina clay by Fredlund (1964) and for Indian Head till by Vanapalli et al. (1996). A good fit was obtained using the unimodal equation with one bending point. The unimodal equation with two bending points was also used for the Indian Head

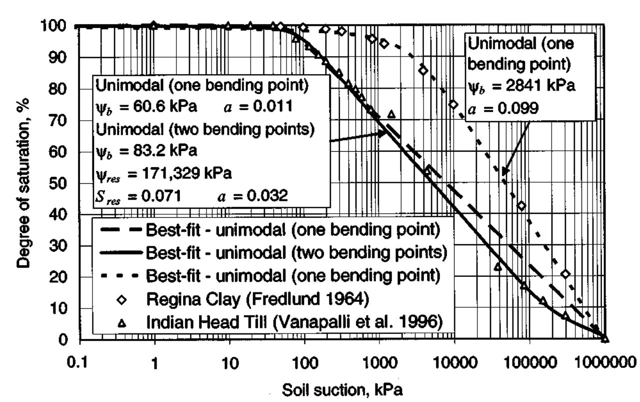

Fig. 5. Best-fit curves to the experimental data of Regina clay and Indian Head till

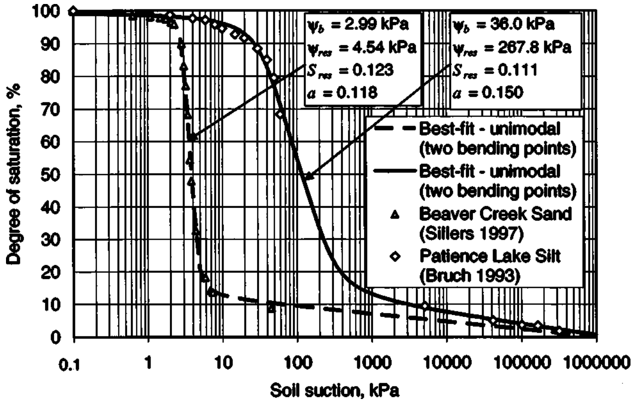

Fig. 6. Best-fit curves to the experimental data of Patience Lake silt and Beaver Creek sand

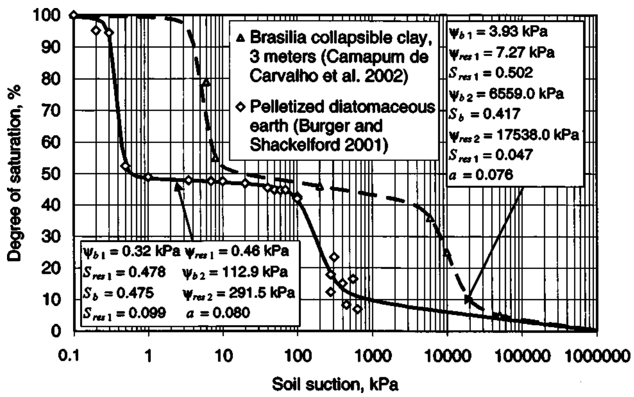

Fig. 7. Best-fit curves to the experimental data of pelletized diatomaceous earth and of the Brasilia collapsible clay

till, giving a slightly better fit. Fig. 6 presents two other data sets, for Patience Lake silt (Bruch 1993) and Beaver Creek sand (Sillers 1997) along with the best-fit curve and its parameters. A close fit is observed.

SWCC experimental data for two bimodal soils was used to demonstrate the fitting capability of the proposed bimodal equation. Fig. 7 shows the best-fit curve, the fitting parameters, and the experimental data for a pelletized diatomaceous earth (Burger and Shackelford 2001) and for a residual, highly collapsible clay from Brasilia (Camapum de Carvalho et al. 2002). Close fits are again observed.

Conclusions

Flexible mathematical representations for both unimodal and bimodal soil-water characteristic curves have been proposed. The proposed equations are defined by parameters that have physical meaning and that are independently related to shape features of the SWCC. Parametric analyses and fitting to experimental data sets were used to illustrate the fitting capability of the proposed equations, with excellent results. The proposed equation can make the treatment of SWCC data easier, and statistical analyses on a large amount of data will benefit from the use of an equation whose parameters are mathematically independent.

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Notation

The following symbols are used in this technical note:

a = hyperbolas sharpness variable;
d, d_1, d_2, d_3 = weight factors;
r_i = hyperbolas aperture angle tangent;
S, S_b = first and second degrees of saturation;
S_i = individual hyperbolas;
S_i^a = y coordinate of the hyperbolas center;
$S_{res}, S_{res1}, S_{res2}$ = residual degrees of saturation;
θ_i = hyperbolas rotation angles;
ψ = soil suction;
$\psi_b, \psi_{b1}, \psi_{b2}$ = air-entry values;
ψ_i^a = x coordinate of the hyperbolas center; and
$\psi_{res}, \psi_{res1}, \psi_{res2}$ = residual soil suctions.

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