

An equation to represent grain-size distribution

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Abstract: The grain-size distribution is commonly used for soil classification; however, there is also potential to use the grain-size distribution as a basis for estimating soil behaviour. For example, much emphasis has recently been placed on the estimation of the soil-water characteristic curve. Many methods proposed in the literature use the grain-size distribution as a starting point to estimate the soil-water characteristic curve. Two mathematical forms are presented to represent grain-size distribution curves, namely, a unimodal form and a bimodal form. The proposed equations provide methods for accurately representing uniform, well-graded soils, and gap-graded soils. The five-parameter unimodal equation provides a closer fit than previous two-parameter, log-normal equations used to fit uniform and well-graded soils. The unimodal equation also improves representation of the silt- and clay-sized portions of the grain-size distribution curve.

Key words: grain-size distribution, sieve analysis, hydrometer analysis, soil classification, probability density function.

Résumé : La distribution granulométrique est utilisée couramment pour la classification des sols; cependant, il est possible d'utiliser également la distribution granulométrique comme base d'évaluation du comportement du sol. Par exemple, beaucoup d'emphase a été mise récemment sur la détermination de la courbe caractéristique sol-eau. Plusieurs méthodes proposées dans la littérature utilisent la distribution granulométrique comme point de départ pour établir la courbe caractéristique sol-eau. Deux formes mathématiques sont présentées pour reproduire les courbes de distribution granulométrique: nommément, une forme unimodale et une forme bimodale. Les équations proposées fournissent des méthodes pour représenter avec précision des sols à granulométrie uniforme, étalée et discontinue. L'équation unimodale à cinq paramètres fournit une meilleure concordance que les équations antérieures log normales à deux paramètres utilisées pour reproduire les courbes des sols à granulométrie uniforme et étalée. L'équation unimodale améliore aussi la représentation des portions de silt et de grosseurs argileuses de la courbe de distribution granulométrique.

Mots clés : distribution granulométrique, analyse par tamisage, analyse à l'hydromètre, classification des sols, fonction de densité probabilistique.

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Introduction

The grain-size distribution is a simple, yet informative test routinely performed in soil mechanics to classify soils. Recent research has made use of the grain-size distribution as a basis for the estimation of other soil properties such as the soil-water characteristic curve through mathematical analysis (Gupta and Larson 1979*a*, 1979*b*; Arya and Paris 1981; Haverkamp and Parlange 1986). Mathematically representing the grain-size distribution provides several benefits. First, the soil may be classified using the best-fit parameters. Second, the mathematical equation can be used as the basis for analysis related to estimating the soil-water characteristic curve. Third, a mathematical equation can provide a method of representing the entire curve between measured data points. Representing the soil as a mathematical function also provides increased flexibility in searching for similar soils in databases.

American Society for Testing and Materials standards D1140-54 and D422-63 (ASTM 1964*a*, 1964*b*) provide a basic testing and reporting method whereby the results of a sieve and hydrometer analysis are plotted on a semilogarithmic graph. An interpretation method for the series of plotted points is specified in the procedure. Manual interpretation methods, such as sketching in a complete curve, have often been used to provide a complete grain-size distribution curve. Gardner (1956) proposed a two-parameter, log-normal distribution to provide representation of grain-size distribution data. Both methods are feasible but have limitations that are discussed later in the paper.

This paper proposes two new models to fit grain-size data, namely, the use of a unimodal and a bimodal mathematical function. The two new equations provide greater flexibility for fitting a wide variety of soils.

Background

Numerous methods have been developed for particle-size analysis in the laboratory and field. These include the elutriation method, the test tube shaking method, the Wiegner sedimentation cylinder, the photoelectric method, the pipette method, and the hydrometer method, in addition to the sieve analysis. Of these methods, only the pipette and

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Table 1. Equations that have been used to represent the soil-water characteristic curve.

Authors	Equation	Definition of variables
Gardner 1958	$w_w = w_{rg} + (w_s - w_{rg}) \left[\frac{1}{[1 + a_g \Psi^{n_g}]^{n_g}} \right]$	w_s , saturated gravimetric water content; w_w , any gravimetric water content; w_{rg} , residual gravimetric water content; a_g , fitting parameter; n_g , fitting parameter; Ψ , soil suction
Brooks and Corey 1964	$w_w = w_r + (w_s - w_r) \left[\frac{a_c}{\Psi} \right]^{n_c}$	a_c , bubbling pressure (kPa); n_c , pore-size index; w_s , saturated gravimetric water content; w_w , any gravimetric water content; w_r , residual volumetric water content; Ψ , soil suction (kPa)
van Genuchten 1980; Burdine 1953	$w_w = w_{rb} + (w_s - w_{rb}) \left[\frac{1}{[1 + (a_b \Psi)^{n_b}]^{1 - \frac{2}{n_b}}} \right]$	w_s , saturated gravimetric water content; w_w , any gravimetric water content; w_{rb} , residual gravimetric water content; a_b , fitting parameter; n_b , fitting parameter; Ψ , soil suction
van Genuchten 1980; Mualem 1976	$w_w = w_{rm} + (w_s - w_{rm}) \left[\frac{1}{[1 + (a_m \Psi)^{n_m}]^{1 - \frac{1}{n_m}}} \right]$	w_s , saturated gravimetric water content; w_w , any gravimetric water content; w_{rm} , residual gravimetric water content; a_m , fitting parameter; n_m , fitting parameter; Ψ , soil suction
van Genuchten 1980	$w_w = w_{rvg} + (w_s - w_{rvg}) \left[\frac{1}{[1 + (a_{vg} \Psi)^{n_{vg}}]^{m_{vg}}} \right]$	w_s , saturated gravimetric water content; w_w , any gravimetric water content; w_{rvg} , residual gravimetric water content; a_{vg} , fitting parameter; n_{vg} , fitting parameter; m_{vg} , fitting parameter; Ψ , soil suction
Fredlund and Xing 1994	$w_w = w_s \left[1 - \frac{\ln \left(1 + \frac{\Psi}{h_r} \right)}{\ln \left(1 + \frac{10^6}{h_r} \right)} \right] \left[\frac{1}{\left\{ \ln \left[\exp(1) + \left(\frac{\Psi}{a_f} \right)^{n_f} \right] \right\}^{m_f}} \right]$	w_s , saturated gravimetric water content; w_w , any gravimetric water content; a_f , fitting parameter closely related to the air-entry value for the soil; n_f , fitting parameter related to the maximum slope of the curve; m_f , fitting parameter related to the curvature of the slope; h_r , parameter used to adjust lower portion of the curve; Ψ , soil suction

hydrometer methods have found general acceptance for fine-grained soils (Kohnke 1968).

ASTM (1964a, 1964b) presents a standard for testing for grain-size distribution. The interpretation of the grain-size distribution is typically carried out manually. Further details concerning the testing procedure and the interpolation of the sieve and hydrometer tests are provided by Lambe (1951).

Gardner (1956) used a two-parameter, log-normal distribution to fit grain-size distribution data. Kemper and Chepil (1965) further studied the work of Gardner. The two-parameter fit of the grain-size distribution was performed using a geometric mean parameter, x_g , and a geometric standard deviation, σ_g . The method of fitting log-normal equations to the grain-size distribution was not recommended for general use. However, the reason given for not using the log-normal method was the lack of computing power necessary to fit the equation to data. Hagen et al. (1987) presented a computerized, iterative procedure that required only two sieves to determine the parameters for a standard, two-parameter log-normal distribution. Unfortunately, the log-normal distribu-

tion often failed to provide a close fit of the grain-size distribution at the extremes of the curve (Gardner 1956; Hagen et al. 1987). Wagner and Ding (1994) later improved upon the log-normal equation by presenting three- and four-parameter log-normal equations.

Campbell (1985) presented a classification diagram based on the assumption that the particle-size distribution is approximately log normal. This assumption led to the particle-size distribution being approximated with a Gaussian distribution function. With this assumption, any combination of sand, silt, and clay can be represented by a geometric (or logarithmic) mean particle diameter and a geometric standard deviation. Values were summarized in a modified U.S. Department of Agriculture (USDA) textural classification chart by Shirizi and Boersma (1984).

The first limitation associated with using a log-normal type of equation is the assumption that the grain-size distribution is symmetric. In reality, the grain-size distribution is often nonsymmetric and can be better fit by a different type of equation. Second, a method for fitting soils that are

bimodal or gap-graded is of value and the four-parameter log-normal equations have not been found to be satisfactory for fitting these types of grain-size distribution.

There are three general categories of grain-size distributions (Holtz and Kovacs 1981): well-graded soils, uniform soils, and gap-graded soils. This paper focuses on these three categories of grain-size distribution and provides equations to fit the experimental data for each category. The well-graded and uniform soils are examined using a unimodal method of fitting an equation, and then a mathematical means of representing a gap-graded soil is presented.

Unimodal equation for grain-size distribution data

The selection of an appropriate, mathematical equation involved a review of a variety of equations that could be used to fit soils data. It has been observed that the soil-water characteristic curve possesses a shape similar to that of the grain-size distribution curve. This is probably to be expected, since the soil-water characteristic provides a representation of the void distribution in a soil, whereas the grain-size curve provides information on the distribution of the solid phase of the soil. Since the solids plus the voids add up to the total soil volume, it is to be expected that the distribution of the solids phase (i.e., grain-size distribution) would tend to bear an inverse relationship to the distribution of voids (i.e., represented by the soil-water characteristic curve), and vice versa.

A summary of several of the equations that have been used to fit the soil-water characteristic curve is given in Table 1. Brooks and Corey (1964) and Gardner (1974) presented three-parameter equations and van Genuchten (1980) and Fredlund and Xing (1994) presented four-parameter equations. It would seem reasonable that a form of equation similar to those shown in Table 1 could be used to represent the grain-size distribution.

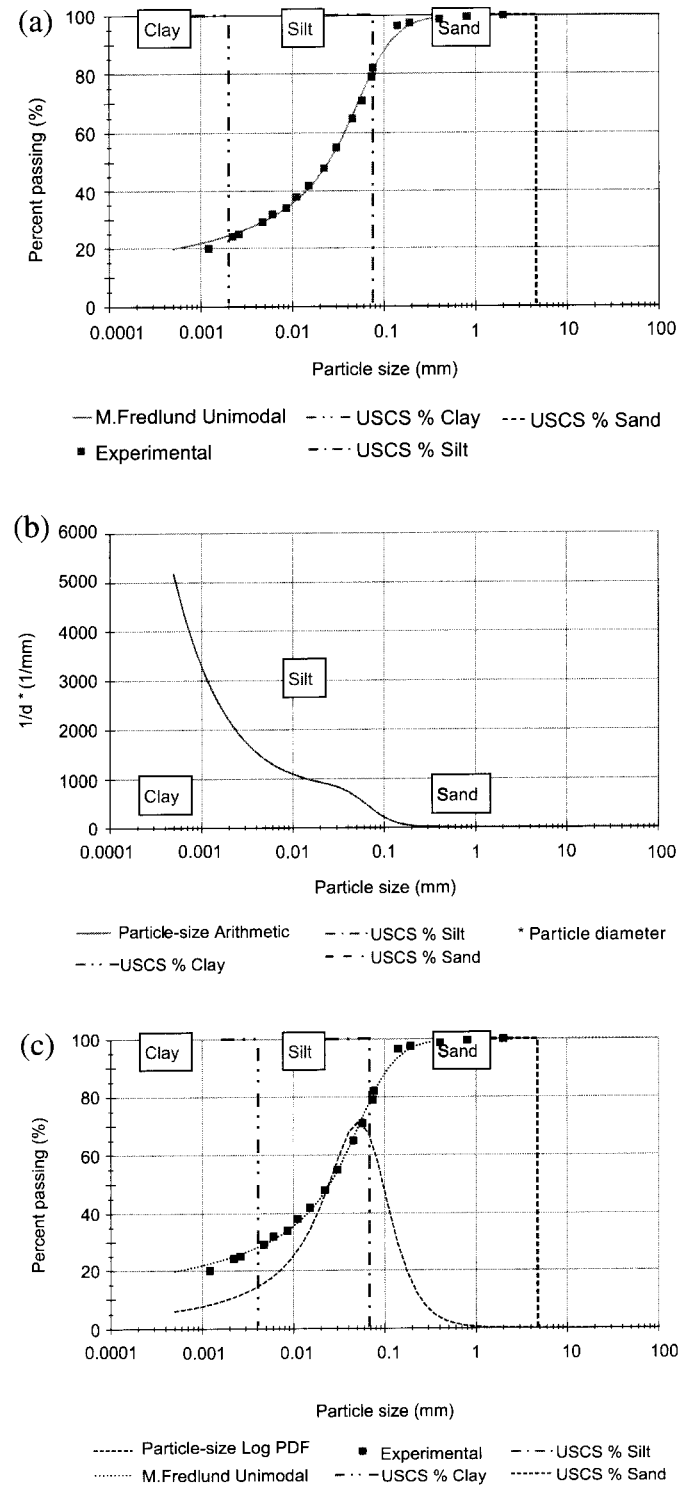
An accurate representation of the clay fraction of the grain-size distribution was considered necessary to complete the mathematical function. Since the Fredlund and Xing (1994) equation allows independent control over the lower end of the curve (i.e., the fine particle size range), it was selected as the basis for the development of a grain-size distribution equation. The reversed scale of the grain-size distribution and characteristics unique to the grain-size distribution required the original Fredlund and Xing equation to be modified to the form shown as follows:

$$[1] \quad P_p(d) = \frac{1}{\left\{ \ln \left[\exp(1) + \left(\frac{a_{gr}}{d} \right)^{n_{gr}} \right] \right\}^{m_{gr}}} \left\{ 1 - \frac{\left[\ln \left(1 + \frac{d_{rgr}}{d} \right) \right]^7}{\left[\ln \left(1 + \frac{d_{rgr}}{d_m} \right) \right]^7} \right\}$$

where

$P_p(d)$ is the percentage, by mass, of particles passing a particular size;

Fig. 1. Grain-size data fit with unimodal equation for a clayey silt: (a) best-fit curve, $R^2 = 0.998$; (b) arithmetic probability density function; (c) logarithmic probability density function (soil number 10030).



a_{gr} is a parameter designating the inflection point on the curve and is related to the initial breaking point on the curve;

n_{gr} is a parameter related to the steepest slope on the curve (i.e., uniformity of the particle-size distribution);

m_{gr} is a parameter related to the shape of the curve as it approaches the fines region;

d_{rgr} is a parameter related to the amount of fines in a soil;

d is the diameter of any particle size under consideration; and

d_m is the diameter of the minimum allowable size particle.

Equation [1] is referred to as a unimodal equation and can be used to fit a wide variety of soils. A quasi-Newton fitting algorithm was used to adjust three of the four parameters to fit the equation to each soil. The algorithm progressively minimizes the squared differences between the equation and experimental data. The best-fit particle-size distribution function can be plotted along with the grain-size distribution data, typically on a logarithmic scale, as shown in Fig. 1a for a clayey silt (soil number 10030).¹

The unimodal equation provides significant improvements in the fit of grain-size data over previous mathematical representations (i.e., log-normal distribution). This is to be expected due to the increase in the number of parameters used to represent the grain-size distribution. The complexity of the proposed unimodal equation due to the added parameters is determined to be insignificant because of the availability of curve-fitting software.

The particle-size distribution provides information on the amount and dominant sizes of particles present in a soil. However, another form can also be used to represent the distribution of particle sizes by differentiating the particle-size distribution curve. The differentiation produces a particle-size probability density function (PDF). The differentiated form of the unimodal grain-size equation is given in eq. [2], and the parameters presented in the particle-size PDF are the same as those defined for eq. [1]:

$$[2] \quad \frac{dP_p}{dd} = \frac{1}{\left\{ \ln \left[\exp(1) + \left(\frac{a_{gr}}{d} \right)^{n_{gr}} \right] \right\}^{m_{gr}}} \left\{ 1 - \left[\frac{\ln \left(1 + \frac{d_{rgr}}{d} \right)}{\ln \left(1 + \frac{d_{rgr}}{d_m} \right)} \right]^7 \right\} m_{gr} \left(\frac{a_{gr}}{d} \right)^{n_{gr}} \\
 \times \frac{n_{gr}}{d \left\{ \left[\exp(1) + \left(\frac{a_{gr}}{d} \right)^{n_{gr}} \right] \ln \left[\exp(1) + \left(\frac{a_{gr}}{d} \right)^{n_{gr}} \right] \right\}} \\
 + \frac{7}{\left\{ \ln \left[\exp(1) + \left(\frac{a_{gr}}{d} \right)^{n_{gr}} \right] \right\}^{m_{gr}}} \frac{\ln \left(1 + \frac{d_{rgr}}{d} \right)^6}{\ln \left(1 + \frac{d_{rgr}}{d_m} \right)^7} \frac{d_{rgr}}{\left[d^2 \left(1 + \frac{d_{rgr}}{d} \right) \right]}$$

The particle-size distributions presented in this paper are calculated using eq. [2] and are referred to as the arithmetic probability density function. Figure 1b illustrates the arithmetic probability density function for the clayey silt (soil number 10030) shown in Fig. 1a.

The highest point in the PDF plot is the mode or the most frequent particle size. Since eq. [2] is a PDF, the natural laws of probability hold, such that the area under the differentiated curve must equal unity:

$$[3] \quad \int_{-\infty}^{+\infty} \left(\frac{dP_p}{dd} \right) dx = 1$$

Equation [2] can be arithmetically integrated between the specified particle-diameter sizes. The probability that a soil particle diameter will fall in a certain range is determined by the following relationship:

$$[4] \quad \text{probability } (d_1 < d < d_2) = \int_{x=d_1}^{x=d_2} p(x) dx$$

It is convenient to represent the PDF in a different manner when plotted on a logarithmic scale. The arithmetic PDF will often appear distorted when plotted on a logarithmic scale. The peak computed from eq. [4] will not represent the most frequent particle size. To overcome this limitation, the PDF is often represented by taking the logarithm of the particle size and differentiating the grain-size equation to produce a PDF which appears more physically realistic as presented in eq. [5]:

$$[5] \quad p_1(d) = \frac{dP_p}{d \log(d)} = \frac{dP_p}{dd} \ln(10)d$$

where $p_1(d)$ is the logarithmic PDF.

The peak of eq. [5] will represent the most frequent particle size. It must be noted that the probability of the logarithmic PDF must be calculated according to eq. [6]:

$$[6] \quad \text{probability } (d_1 < d < d_2) = \int_{x=\log(d_1)}^{x=\log(d_2)} p_1(x) dx$$

Figure 1c shows the logarithmic PDF for the clayey silt (soil number 10030).

The unimodal equation fit for a silty sand (soil number 63) and a sandy clay (soil number 11648) are shown in Figs. 2a and 3a, respectively. Also shown are the arithmetic probability density functions (i.e., Figs. 2b, 3b) and the logarithmic probability density function for each of the above soils (i.e., Figs. 2c, 3c).

The variation of R^2 (where R is the correlation coefficient) versus percent clay is shown in Fig. 4. The value of R^2 was plotted versus percent clay because the representation of fines by the new equation was considered important.

¹ Soil numbers refer to soils found in the SoilVision database, which is a proprietary product of SoilVision Systems Ltd.