ABSTRACT: Practical applications of unsaturated soil mechanics still lag behind the state-of-the-art knowledge. The main stumbling block is the time-consuming processes involved in the measurement of the unsaturated soil parameters required for the constitutive models. Recent research has shown that the soil-water characteristic curves of a soil can be used in the establishment of a number of the unsaturated soil parameters. In many applications it has become obvious that a satisfactory equation for describing the soil-water characteristic curve will simplify the determination of the soil parameters. Over the years a number of equations have been suggested. Most of these equations have limited success depending on soil types. This paper evaluates the more popular equations that have been suggested and shows that all the equations can be derived from a single generic form. One equation has been identified that performs very well for all soil types. If this equation is in common usage, useful databases on unsaturated soil parameters can be more easily established for practical applications of unsaturated soil mechanics.

INTRODUCTION

Fredlund and Morgenstern (1977) concluded that net normal stress and matric suction are the stress state variables of an unsaturated soil. The water content in an unsaturated soil is a function of the suction present in the soil. This relationship between the water content in a soil and the suction can be expressed in a plot of volumetric water content versus suction that is known as the soil-water characteristic curve. This curve is more commonly referred to as a soil-water retention curve in soil sciences. The soil-water characteristic curve of a soil can be obtained using a pressure plate device in the laboratory. Using the axis-translation technique (Hilf 1956), air pressure above atmospheric is applied to the soil specimen while the water pressure is kept at a lower value that is usually atmospheric. This is made possible through the use of a high-air entry disk that separates the air phase from the water phase. The difference between the air and water pressures is known as matric suction. The water content of the soil specimen at various matric suctions can therefore be determined and a soil-water characteristic curve is obtained.

Vanapalli et al. (1996) presented a relationship between the soil-water characteristic curve and saturated shear strength parameters. The more established usage of the soil-water characteristic curve is the derivation of permeability function from the curve (Millington and Quirk 1961; Mualem 1976). The soil-water characteristic curve is also required in the determination of water volume changes in the soil with respect to matric suction changes. The coefficient of water volume change with respect to matric suction is given by the slope of the soil-water characteristic curve. For these applications it is more useful if the soil-water characteristic curve can be expressed as an equation. Over the years a number of equations have been suggested for the soil-water characteristic curve. This paper evaluates various forms of equations for different soil types.

SOIL-WATER CHARACTERISTIC CURVE EQUATIONS

Typical soil-water characteristic curves for a sandy soil, a silty soil, and a clayey soil are shown in Fig. 1. From the soil-water characteristic curve a few parameters can be defined: the saturated volumetric water content, \( \theta_s \), the residual volumetric water content, \( \theta_r \), the air-entry value or bubbling pressure, \( \psi_b \), and the residual air content, \( \theta_a \) (Fig. 2). A number of equations have been suggested for the soil-water characteristic curve and almost all the equations suggested can be derived from the following generic form:

\[
\theta_s \psi^2 + a_2 \exp(a_3 \psi^2) = a_4 \psi^2 + a_5 \exp(a_6 \psi^2) + a_7
\]  

where \( a_1, a_2, a_3, a_4, a_5, a_6, a_7, b_1, \) and \( b_2 \) are constants; \( \psi \) = suction pressure; and \( \theta = \text{normalized volumetric water content} \), i.e., \( (\theta - \theta_r)/\theta_i - \theta_r \) where \( \theta_i \) = volumetric water content.

FIG. 1. Soil-Water Characteristics Curves for Sandy Soil, Silty Soil, and Clayey Soil (after Fredlund and Xing 1994)

FIG. 2. Definitions of Terms of Typical Soil-Water Characteristics Curve (from Fredlund and Xing 1994)
If \( a_2 = a_5 = a_7 = 0 \) and \( b_1 = 1 \), then (1) can be simplified as

\[
\Theta = \frac{a_2}{a_1} \psi^b.
\]  
(2)

By letting \( b_2 = -\lambda \) and \( a_4/a_1 = \psi_0^b \) in (2), the Brooks and Corey (1964) equation for soil-water characteristic curve is obtained

\[
\Theta = \left( \frac{\psi}{\psi_0} \right)^{-\lambda}.
\]  
(3)

If the natural logarithms of both sides of (2) is equated, the following equation is obtained:

\[
\ln \Theta = A + B \ln \psi
\]  
(4)

where \( A = \ln(a_4/a_1) \) and \( B = b_2 \).

Eq. (4) is the form used by Williams et al. (1983) to describe the soil-water characteristic curve of many Australian soils.

If \( a_2, a_4, \) and \( a_7 \) are set to 0 and \( b_2 \) and \( b_1 \) are set to 1 in (1), the following equation is obtained:

\[
\Theta = \frac{a_2}{a_1} \exp(a_4 \psi)
\]  
(5a)

In (5a), \( a_3/a_1 \) can be written as \( A \exp(-B) \), where \( A \) and \( B \) are constants, to give

\[
\Theta = A \exp(a_4 \psi - B)
\]  
(5b)

which is the exponential function suggested by McKee and Bumb (1984) and has been referred to as the Boltzmann distribution.

If \( a_2 = a_4 = 0, a_1 = a_2, b_1 = -1, \) and \( b_2 = 1 \) in (1) and again writing \( a_3/a_1 \) as \( A \exp(-B) \), the following relationship that was subsequently suggested by McKee and Bumb (1987) to improve the fit of (5b) at or near fully saturated conditions is obtained

\[
\Theta = \frac{1}{1 + A \exp(a_4 \psi - B)}
\]  
(6)

If we let \( a_2, a_4 \) be 0 and \( a_1 = a_2 \) in (1), the following equation is obtained:

\[
\Theta = \left( \frac{a_2}{a_1} \psi^b + 1 \right)^{-1}
\]  
(7)

By letting \( a_4/a_1 = a, b_1 = 1, \) and \( b_2 = n \) in (7), the more familiar Gardner (1958) equation is obtained

\[
\Theta = \frac{1}{1 + a \psi^{b_2}}
\]  
(8)

where \( a \) and \( n \) are constants.

However, if \( a_4/a_1 = a^*, b_1 = m, \) and \( b_2 = n \) in (7), the van Genuchten (1980) equation is obtained

\[
\Theta = \left[ \frac{1}{1 + (\alpha \psi^{b_2})} \right]^m
\]  
(9)

where \( \alpha, n, \) and \( m \) are constants.

In (1), if \( a_1 \) and \( a_2 \) are set equal to 0 and \( a_3 \) is set to 1, the following equation is obtained

\[
\Theta^b = \ln \left( \frac{a_2}{a_1} + \frac{a_4}{a_2} \psi^b \right)
\]  
(10)

The following equation suggested by Fredlund and Xing (1994) can be obtained by substituting \( a_4/a_1 = e, a_4/a_2 = (1/e)^b, b_1 = m, \) and \( b_2 = n \) into (10):

\[
\Theta = \frac{1}{\left[ \ln \left( e + \frac{\psi^n}{a} \right) \right]^m}
\]  
(11)

where \( a, n, \) and \( m \) are constants and \( e \) is the natural base of logarithms.

If in (1), \( a_1, a_2, \) and \( a_3 \) are set to 0 and \( b_1 \) and \( b_2 \) are set to 1, the following equation is obtained

\[
\exp(a_4 \Theta) = \frac{a_2}{a_1} \psi
\]  
(12)

Eq. (12) can be transformed into the following equation as suggested by Farrel and Larson (1972):

\[
\psi = \psi_0 \exp(a(1 - \Theta))
\]  
(13)

where \( \psi_0 \) is the air-entry value and \( a \) is a constant.

Most of the soil-water characteristic curve equations described earlier are empirical in nature. The equations were suggested based on the shape of the soil-water characteristic curve. From Fig. 1 it may be observed that the general shape of the soil-water characteristic curve is sigmoidal. Some of the equations listed in the foregoing do not give a sigmoid curve. These include the equations of Gardner (1958), Brooks and Corey (1964), Farrel and Larson (1972), Williams et al. (1983), and McKee and Bumb (1984). The shapes of these equations are illustrated in Fig. 3. The equations that provide a sigmoid curve are the equations of van Genuchten (1980), McKee and Bumb (1987), and Fredlund and Xing (1994). The shapes of these equations are illustrated in Fig. 4.
\( \theta_v(R) = \int_{R_{\text{min}}}^{R} f(r) \, dr \) \hspace{1cm} (14)

\( \theta_v(R) \) = volumetric water content when all the pores with radius less than or equal to \( R \) are filled with water; and \( R_{\text{min}} \) = minimum pore radius in the soil. It was shown that the Brooks and Corey (1964) equation is valid only when the pore size distribution is close to the distribution \( f(R) = A R^{m+1} \) where \( A \) and \( m \) are constants. It was also shown that the McKee and Bumb (1984) equation given by (5) is valid when the pore size distribution of the soil is close to a gamma distribution. The pore-size distribution function suggested by Fredlund and Xing (1994) [(11)] is a modification of the pore-size distrib-

\( \text{FIG. 5. Effect of } a, n, \text{ and } m \text{ on Shapes of van Genuchten and Fredlund and Xing Soil-Water Characteristics Equations} \)
\[ C(\psi) = 1 - \frac{1}{\ln \left( \frac{1 + \psi}{\psi_r} \right)} \]  
(16)

where \( \psi \) = suction value that corresponds to the residual volumetric water content \( \theta_r \). The choice of a suction value of 1,000,000 kPa in (16) is based on experimental evidence that the volumetric water content in soils approaches zero as the suction tends to 1,000,000 kPa (Coleman and Croney 1961; Koorevaar et al. 1983). This suction value is also supported by thermodynamic considerations. The thermodynamic relationship between soil suction and the partial pressure of pore-water vapor is given as follows:

### TABLE 1. Minimum Sum of Squared Residual Values (SSR) for Three- and Four-Parameter Equations

<table>
<thead>
<tr>
<th>Soil type</th>
<th>Three-Parameter Equations</th>
<th>Four-Parameter Equations</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Gardner (1980) equation 20</td>
<td>Fredlund and Xing (1994) equation 24</td>
</tr>
<tr>
<td>Lakeland sand</td>
<td>0.0000251 (3)</td>
<td>0.0000189 (3)</td>
</tr>
<tr>
<td>Superstition sand</td>
<td>0.00014 (4)</td>
<td>0.000139 (2)</td>
</tr>
<tr>
<td>Mine tailings</td>
<td>0.000337 (6)</td>
<td>0.000144 (1)</td>
</tr>
<tr>
<td>Columbia sandy loam</td>
<td>0.000205 (5)</td>
<td>0.001864 (4)</td>
</tr>
<tr>
<td>Drummer soil 0–30 cm</td>
<td>0.000014 (5)</td>
<td>0.000105 (2)</td>
</tr>
<tr>
<td>Drummer soil 30–75 cm</td>
<td>0.0000339 (5)</td>
<td>0.0000878 (1)</td>
</tr>
<tr>
<td>Drummer soil 75–90 cm</td>
<td>0.0000545 (5)</td>
<td>0.000239 (2)</td>
</tr>
<tr>
<td>Touchet silt loam (GE3)</td>
<td>0.008804 (5)</td>
<td>0.001597 (4)</td>
</tr>
<tr>
<td>Guelph loam (dry- ing)</td>
<td>0.0008985 (6)</td>
<td>0.000158 (1)</td>
</tr>
<tr>
<td>Yolo light clay</td>
<td>0.0002425 (5)</td>
<td>0.0001774 (1)</td>
</tr>
<tr>
<td>Beit Netofa clay</td>
<td>0.0000757 (5)</td>
<td>0.0033072 (2)</td>
</tr>
<tr>
<td>Total weightage</td>
<td>54</td>
<td>23</td>
</tr>
</tbody>
</table>

Note: Values in parentheses are the weightage where a value of 1 indicates least SSR value and a value of 6 indicates largest SSR value for data set.

### TABLE 2. Fitted Parameters for Three- and Four-Parameter Equations

<table>
<thead>
<tr>
<th>Soil type</th>
<th>Three-Parameter Equations</th>
<th>Four-Parameter Equations</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Gardner (1968) equation 20</td>
<td>Fredlund and Xing (1994) equation 24</td>
</tr>
<tr>
<td>Lakeland sand</td>
<td>0.127, 43.8, 2.59</td>
<td>0.1229, 2.84, 3.80, 4.10</td>
</tr>
<tr>
<td>Superstition sand</td>
<td>0.289, 216, 4.03</td>
<td>0.267, 2.87, 5.87, 4.01</td>
</tr>
<tr>
<td>Mine tailings</td>
<td>0, 2.056, 3.06</td>
<td>0.1055, 55.8, 3.40, 18.60</td>
</tr>
<tr>
<td>Columbia sandy loam</td>
<td>0.520, 1.406, 8.58</td>
<td>0.436, 5.98, 10.68, 0.354</td>
</tr>
<tr>
<td>Drummer soil 0–30 cm</td>
<td>0.320, 611, 1.321</td>
<td>0.242, 3.03, 0.264</td>
</tr>
<tr>
<td>Drummer soil 30–75 cm</td>
<td>0.323, 330, 1.519</td>
<td>0.282, 14.43, 4.44, 0.299</td>
</tr>
<tr>
<td>Drummer soil 75–90 cm</td>
<td>0.341, 135.3, 1.999</td>
<td>4.9, 7.3, 12.9, 13.75, 0.335</td>
</tr>
<tr>
<td>Touchet silt loam (GE3)</td>
<td>0.322, 1.942, 4.21</td>
<td>0.323, 6.98, 10.68, 0.1503</td>
</tr>
<tr>
<td>Guelph loam (dry- ing)</td>
<td>0, 1.172, 0.262</td>
<td>0.1643, 5.93, 2.10, 0.273</td>
</tr>
<tr>
<td>Yolo light clay</td>
<td>0.490, 20.4, 1.538</td>
<td>0.387, 2.82, 2.41, 0.238</td>
</tr>
<tr>
<td>Beit Netofa clay</td>
<td>0.0848, 102.3, 0.683</td>
<td>0.2810, 0.746, 0.393</td>
</tr>
</tbody>
</table>

Note: Values in parentheses are the weightage where a value of 1 indicates least SSR value and a value of 6 indicates largest SSR value for data set.
where \( \psi = \) soil suction or total suction in kPa; \( R = \) universal gas constant \([=8.31432 \text{ J/(mol K)}]\); \( T = \) absolute temperature in K; \( v_o = \) specific volume; \( \omega_v = \) molecular mass of water vapor \([=18.016 \text{ kg/mol}]\); \( a_v = \) partial pressure of pore-water vapor in kPa; and \( a_{v_0} = \) saturation pressure of water vapor over a flat surface of pure water at the same temperature. The ratio of \( a_v/a_{v_0} \) is called relative humidity. At a temperature of 20°C and a relative humidity of 0.01%, (17) gives \( \psi \) as 1,026,289 kPa. However, it is to be noted that with the introduction of \( \psi \), the accompanying pore-size distribution function that Fredlund and Xing (1994) assumed should become more complicated than suggested.

### EVALUATION OF SOIL-WATER CHARACTERISTIC CURVE FUNCTIONS

The soil-water characteristic equation listed in the foregoing involves unknown parameters that have to be determined. In the preceding section it has been shown that the soil-water characteristic equations can be derived from a generic equation involving seven parameters. Most of the time the saturated volumetric water content \( \theta_s \) is determined whereas the residual volumetric water content \( \theta_r \) is not always determined. To date, the maximum number of parameters suggested in soil-water characteristic equations is four if \( \theta_r \) is treated as a known. In the following section, the soil-water characteristic equations will be grouped into the number of curve-fit parameters that have to be determined (for ease of discussion, the unknown parameters are labeled as \( a, b, \) and \( c \)).

The two-parameter equation given by Williams et al. (1983) is

\[
\ln \psi = a + b \ln \theta_w \quad \text{(unknowns:} \ a, b) \tag{18}
\]

The three-parameter equations are

1. Gardner (1958):

   \[
   \theta_w = \theta_r + \frac{\theta_s - \theta_r}{1 + a^b} \quad \text{(unknowns:} \ \theta_r, a, \text{ and } b) \tag{19}
   \]

   Eq. (19) is more attractive if expressed in the following form as \( a \) will have the same units as \( \psi \), and \( b \) will then be independent of units:

   \[
   \theta_w = \theta_r + \frac{\theta_s - \theta_r}{1 + \alpha \psi^b} \tag{20}
   \]

   The preceding equation is more commonly referred to as a logistic curve (Seber and Wild 1989) where \( a = \) matric suction value that corresponds to a volumetric water content of \( (\theta_s + \theta_r)/2 \); and \( b = \) slope factor.

2. Brooks and Corey (1964):

   \[
   \theta_w = \theta_r + (\theta_s - \theta_r) \left( \frac{\psi + \alpha}{\psi + 1} \right) \quad \text{(unknowns:} \ \theta_r, a, \text{ and } b) \tag{21}
   \]

   Eq. (21) is valid only for \( \psi \) greater than or equal to \( a \) (air-entry value). For \( \psi \) less than \( a \), \( \theta_w \) is equal to \( \theta_r \). For larger values of \( \psi \), (21) will give similar values as (20).

\[ \theta_w = \theta_s + (\theta_s - \theta_n) \exp \left( \frac{a - \psi}{b} \right) \]

(unknowns: \( \theta_n, a, \) and \( b \)) (22)


\[ \theta_w = \theta_s + \frac{\theta_s - \theta_n}{1 + \exp \left( \frac{\psi - a}{b} \right)} \]

(unknowns: \( \theta_n, a, \) and \( b \)) (23)

5. Fredlund and Xing (1994) with correction factor \( C(\psi) = 1 \):

\[ \theta_w = \theta_s + \frac{\theta_s - \theta_n}{1 + (1 + \alpha \psi)^c} \]

(unknowns: \( \theta_n, a, b, \) and \( c \)) (24)

Fredlund and Xing (1994) had mentioned that \( C(\psi) \) is approximately equal to 1 at low suctions as the curve at the low-suction range is not significantly affected by \( C(\psi) \). With \( C(\psi) = 1 \), \( \theta_w \) is not zero when \( \psi \) is 1,000,000 kPa.

The four-parameter equations are

1. van Genuchten (1980):

\[ \theta_w = \theta_s + \frac{\theta_s - \theta_n}{1 + \alpha \psi} \]

(unknowns: \( \theta_n, a, \) and \( b \)) (25)

Similar to (19), (25) is more attractive when expressed as

\[ \theta_w = \theta_s + \frac{\theta_s - \theta_n}{1 + (\psi/a)^b} \]

(unknowns: \( \theta_n, a, b, \) and \( c \)) (26)

thus, \( b \) and \( c \) are now independent of units. The effect of the parameters \( a, b, \) and \( c \) on the shape of the curve is illustrated in Fig. 5 where \( a = a, b = n, \) and \( c = m \). The slope factor, \( b \) (=\( n \)), changes the slope about pivot point \( a (=a) \) and \( c (=m) \) rotates the sloping portion of the curve and the lower plateau at a point above the "knee" of the curve. In (26), \( a \) no longer corresponds to the matric suction at the volumetric water content of \( (\theta_s + \theta_n)/2 \). However, \( a \) can be expressed as a function of this matric suction that we shall denote as \( \psi_{50} \). Substituting, \( \theta_w = (\theta_s + \theta_n)/2 \) and \( \psi = \psi_{50} \) into (26)

\[ a = \frac{\psi_{50}}{(2^{1/\alpha} - 1)^{1/\alpha}} \]

(27)

Depending on the values of \( b \) and \( c \), \( a \) can be greater than or less than \( \psi_{50} \). If \( c = 1 \) as in the Gardner (1958) equation, \( a \) will be equal to \( \psi_{50} \). A similar equation can be obtained for Fredlund and Xing (1994) equation, (24), as follows:

\[ a = \frac{\psi_{50}}{[\exp(2^{1/\alpha}) - 1]^{1/\alpha}} \]

(28)

It is therefore clear from the discussions that the matric suction value \( a \) in (20), (24), and (26) is not the air-entry value or bubbling pressure and should not be interpreted as such.

2. Fredlund and Xing (1994):

\[ \theta_w = \left[ 1 - \frac{\ln \left( \frac{1 + \psi}{\psi_s} \right)}{\ln \left( \frac{1 + 1,000,000}{\psi_s} \right)} \right] \frac{\theta_s}{\left( \ln \left[ e + \left( \frac{\psi}{\psi_s} \right)^b \right] \right)^c} \]

(unknowns: \( \psi_s, a, b, \) and \( c \)) (29)

3. Fredlund and Xing (1994) suggested the use of the fol-
lowing equation if the residual water content \( \theta_r \) is required:

\[
\theta_r = \theta_i + \frac{\theta_s - \theta_i}{\left( \ln \left( \frac{\theta_i}{\theta_s} \right) \right)^b}
\]

(unknowns: \( \theta_i, a, b, \) and \( c \))

(30)

In the preceding equations, the equations that give a sigmoid curve are definitely more versatile and will give a better fit to the soil-water characteristic curve. Therefore, only (20) from researchers had suggested graphical procedures in obtaining parameters for the equations, for example van Genuchten (1980) and Fredlund and Xing (1994). With today's computational advancement, these parameters are best obtained using a minimization algorithm. The quantity to minimize is the sum of the squared residuals, SSR

\[
SSR = \sum_{i=1}^{N} w_i (\theta_i - \theta_j)^2
\]

(31)

where \( w_i \) is a weighting factor. An equal weighting factor is chosen in this paper.

Published soil-water characteristic data for soil materials ranging from sand to clay (Table 1) are used to evaluate the equations. The textural class for the soil types according to the United States Department of Agriculture (USDA) soil classification are the following: Lakeland sand—fine sand; Superstition sand—sand; mine tailings—fine sand; Columbia sandy loam—sandy loam; Drummer soil 0–30 cm—silty loam; Drummer soil 30–75 cm—silty loam; Drummer soil 75–90 cm—silt; Touchet silt loam (GE3)—silty loam; Guelph loam—loam; Yolo light clay—silty clay; and Beit Netofa clay—clay. The Lakeland sand and Drummer soil data are from Elzeftawy and Cartwright (1981), the Superstition sand data are from Richards (1952), the mine tailings data are from Gonzalez and Adams (1980), the Columbia sandy loam and Touchet silt loam (GE3) data are from Brooks and Corey (1964) as referenced by Fredlund et al. (1994), the Guelph loam data are from Elrick and Bowman (1964), the Yolo light clay data are from Moore (1939), and the Beit Netofa clay data are from van Genuchten (1980). Curve fit is performed for the equations using the solver routine provided in the Microsoft Excel software. The best fit for each curve is the one that gave the minimum sum of the squared residual values for the equation. The SSRs are shown in Table 1 and the fitted parameters are shown in Table 2.

The fitted curves for Touchet silt loam (GE3), Yolo light clay, and Beit Netofa clay are shown in Fig. 6. Except for the McKee and Bumb equation (23), all the other equations seem to give a satisfactory fit to the data. To assist the evaluation, a weightage is given to the fit with a weightage of 1 being given to the equation that gave the smallest minimum sum squared residual value (best fit) and a weightage of 6 being given to the equation that gave the largest minimum sum squared residual value (worst fit). The weightages of 1 to 6 are chosen as there are six equations [(20), (23), (24), (26), (29), and (30)] for comparison. The weightage is given in parenthesis in Table 1.

From the total weightage in Table 1 it can be observed that all the four-parameter equations performed much better than the three-parameter equations. Among the three-parameter equations, Fredlund and Xing (24) performs much better than Gardner (20) and McKee and Bumb (23). Among the four-parameter equations, Fredlund and Xing (29) performs marginally better than van Genuchten (26) and Fredlund and Xing (30). Another interesting observation for the four-parameter equations is that for sandy soils (Lakeland sand, Superstition sand, and mine tailings) the equations with the \( \psi_r \) term [(26) and (30)] performed much better than Fredlund and Xing (29). If the sandy soils were left out from the total weightage computations, the total weightage for van Genuchten (26), Fredlund and Xing (29), and Fredlund and Xing (30) will be 21, 9, and 19, respectively in Table 1. In fact if the sandy soils were left out from the total weightage computations, the three-parameter Fredlund and Xing (24) has a total weightage of 21, which is comparable to van Genuchten (26) (total weightage = 21). Gardner (20) and McKee and Bumb (23) will not be pursued further as they are clearly inferior to the others.

The role of \( \psi_r \) in Fredlund and Xing (29) warrants further evaluation. The \( \psi_r \) parameter has been defined by Fredlund and Xing (1994) as the suction corresponding to the residual volumetric water content \( \theta_r \). Furthermore, the effect of \( \psi_r \) on the curve is insignificant at the low-suction range. Plots of (29) with \( a = 300 \, \text{kPa}, \quad b = 10, \quad c = 0.5 \) and three values of \( \psi_r \) (3,000 kPa, 300 kPa, and 30 kPa) are shown in Fig. 7. This figure shows that \( \psi_r \) does affect the initial portion of the curve. The curve with \( \psi_r = 30 \, \text{kPa} \) shows the possibility of obtaining
values of \( \psi \), that violates its definition for being less than \( a \), which in this example is 300 kPa.

The introduction of the correction factor \( C(\psi) \) by Fredlund and Xing (1994), which is a way of forcing the volumetric water content to be zero at high suction, namely 1,000,000 kPa, has no theoretical basis. The writers of the present paper have found two other forms of the "correction" factor \( C(\psi) \) that serve the same purpose. These are

\[
C'(\psi) = -\frac{\ln \left( 2 - \frac{\psi}{A} \right)}{\ln 2}
\]

(32)

and

\[
C''(\psi) = \left( 1 - \frac{\psi}{1,000,000} \right)^a
\]

(33)

where \( A \) and \( B \) are constants. In (32), \( A \) is a \( \psi \), equivalent and the equation suggests that the volumetric water content can be forced to zero at any values of \( A \). Eq. (33) avoids the use of \( \psi \), altogether and thus avoids the problem discussed earlier with \( C(\psi) \).

In any nonlinear curve fit it may be possible to get a few parameter-combinations that produce the same curve. This is highly undesirable in a soil-water characteristic curve equation as it means that its parameters may be very sensitive to the same data when subsets of the data are used to determine the constants. The van Genuchten (26) and Fredlund and Xing (24), (29), and (30) are evaluated in this respect using the mine tailings data from Gonzalez and Adams (1980). The mine tailings data are chosen for their regular intervals and completeness. Three subsets of the mine tailings data, where each contains half as many data points as the previous subset, are used in the evaluation (Fig. 8). The fitted parameters with their var-
FIG. 11. Subsets of Mine Tailings Data to Test Sensitivity of Curve Fit Parameters

Variations as compared to those obtained from the complete data set in parenthesis are shown in Table 3. The fitted curves for each equation are shown in Fig. 9. In general, the different parameters in each equation do not vary the shape of the curves; however, the parameters show large variations in value depending on the number of data points used for the curve fit (Table 3). For the four-parameter equations, van Genuchten (26) and Fredlund and Xing (30) showed large variation in parameters \( a \) and \( c \) as the number of data points are reduced. Fredlund and Xing (29) showed small variations except for the case with six data points where \( \psi \) varies by +33.4%. The parameters in Fredlund and Xing (24) showed small variations particularly parameter \( b \), the slope factor, when compared to (29). The effect of the variation of each parameter on the \( \theta_w \)

value can be evaluated by taking the partial differential of (24), (26), (29), and (30) with respect to each parameter as follows:

For (24):

\[
\frac{d\theta_w}{d\psi} = \frac{\partial \theta_w}{\partial a} da + \frac{\partial \theta_w}{\partial b} db + \frac{\partial \theta_w}{\partial c} dc
\]

(34)

For (26) and (30):

\[
\frac{d\theta_w}{d\psi} = \frac{\partial \theta_w}{\partial a} da + \frac{\partial \theta_w}{\partial b} db + \frac{\partial \theta_w}{\partial c} dc
\]

(35)

For (29):

\[
\frac{d\theta_w}{d\psi} = \frac{\partial \theta_w}{\partial a} da + \frac{\partial \theta_w}{\partial b} db + \frac{\partial \theta_w}{\partial c} dc
\]

(36)
The change in $\theta_{1}$, $d\theta_{1}$, as affected by changes in the values of the parameters for each equation (Table 3) can be observed in Fig. 10. It can be seen that $d\theta_{1}$ is small initially and increases to an approximately constant value. The maximum absolute $d\theta_{1}$ values are 0.6%, 5%, 0.62%, and 5% for (24), (26), (29), and (30), respectively. The worst case for Fredlund and Xing (29) is associated with the data subset of 11 data points and not with the data subset of six data points. In summary, it can be concluded that the fitted parameters are not sensitive to the number of data points as long as there are sufficient points to describe the entire soil-water characteristic curve.

It is generally agreed that curve fitting parameters should be obtained from experimental data that should include points beyond $\theta_{1}$. The effect of incomplete experimental data was investigated using the mine tailings data. The original data set has data at large matric suctions. These data are then reduced by five data points in each subsequent data subsets as shown in Fig. 11. The fitted parameters are shown in Table 4 and the fitted curves for each equation are in Fig. 12. From Fig. 12 it may be observed that the fitted curves coincide with the data points except in the $\theta_{1}$ region, that is, the region of high matric suction where $\theta_{1}$ shows minimal variation. From Table 4 it may be observed that the values of the parameters showed large variations for the data sets having 31 and 26 data points regardless of the equation. For the data set with 36 data points, the equations with the $\theta_{1}$ term [(24) and (30)] showed larger variations when compared to (26) and (29). Worst still, there is no discernible trend in the change in the parameters. Eq. (24) showed the smallest variations for the data set with 36 data points. Generally, if data points after $\theta_{1}$ are not included, the curve starts to deviate. Similar to the earlier discussions, the sensitivity of $\theta_{1}$ to the variations in values of the parameters (Table 4) can be examined using (34)–(36). The changes in $\theta_{1}$, $d\theta_{1}$, can be observed in Fig. 13. The absolute value of $d\theta_{1}$ increases to a constant value. The absolute value of $d\theta_{1}$ for the worst case, which corresponds to 26 data points, increases in the order of Fredlund and Xing (29) (21.5%), Fredlund and Xing (24) (29.5%), van Genuchten (26) (41%), and Fredlund and Xing (30) (100%). Comparing Figs. 10 and 13 leads to the conclusion that it is important to include data points after $\theta_{1}$.

### CONCLUSIONS

The more popular equations for the soil-water characteristic curve have been examined. It has been shown that all the equations reviewed can be described by a generic equation with seven parameters. Some equations are a variant of another. It has also been shown that the parameter $a$ in Gardner (20), van Genuchten (26), and Fredlund and Xing [(24), (29), and (30)] equations is not the air-entry value of the soil as commonly construed. The equation suggested by Fredlund and Xing (1994) (29) gave the best fit among the equations. However, the $\psi$ term in the correction factor $C(\psi)$ affects the initial portion of the soil-water characteristic curve and $\psi$ should not be interpreted as the matric suction corresponding to the residual volumetric water content $\theta_{r}$. Sensitivity analyses tend to favor the use of the Fredlund and Xing equations with the correction factor $C(\psi) = 1$, that is, (24). Another advantage of this equation is that it has only three parameters and so the computational effort in determining the parameters is less than for (26), (29), and (30). It is therefore recommended that the Fredlund and Xing (24) be used for the soil-water characteristic curve. However in obtaining the parameters, the data used should include points after $\theta_{1}$.

### APPENDIX I. DETAILS OF (34)–(36)

#### 1. Fredlund and Xing (24)

\[
\theta_{1} = \left( \ln \left( e + \left( \frac{\psi}{a} \right)^b \right) \right)^c \quad \text{(unknowns: } a, b, \text{ and } c) \]

\[
\frac{d\theta_{1}}{da} = -c\theta_{1} \left( \frac{b \psi}{a} \right) \left( \frac{\psi}{a} \right)^b \ln \left( e + \left( \frac{\psi}{a} \right)^b \right) \]

\[
\frac{d\theta_{1}}{db} = -c\theta_{1} \left( \frac{\psi}{a} \right)^b \ln \left( e + \left( \frac{\psi}{a} \right)^b \right) \]

\[
\frac{d\theta_{1}}{d\psi} = \frac{1}{e + \left( \frac{\psi}{a} \right)^b} \ln \left( e + \left( \frac{\psi}{a} \right)^b \right) \]
FIG. 13. \( \theta_w \) as Affected by Variations in Parameters of (24), (26), (29), and (30) for Data Subsets of Fig. 11

2. van Genuchten (26)

\[ \theta_w = \theta_s \left[ 1 + \left( \frac{\psi}{\alpha} \right)^b \right]^{-1} \]  
(unknowns: \( \theta_s, \alpha, b, \) and \( c \))

\[ \frac{\partial \theta_w}{\partial c} = -\theta_w \ln \left[ e + \left( \frac{\psi}{\alpha} \right)^b \right] \]  
(39)

\[ \frac{\partial \theta_w}{\partial a} = \frac{1 - \left( \frac{\psi}{\alpha} \right)^b}{1 + \left( \frac{\psi}{\alpha} \right)^b} \] 
(40)

\[ \frac{\partial \theta_w}{\partial b} = \frac{(\theta_w - \theta_s) \left( \frac{bc}{a} \right) \left( \frac{\psi}{\alpha} \right)^b}{1 + \left( \frac{\psi}{\alpha} \right)^b} \]  
(41)

3. Fredlund and Xing (29)

\[ \theta_w = \theta_s \left[ 1 - \frac{\ln \left( 1 + \frac{\psi}{\psi_r} \right)}{\ln \left( 1 + \frac{1,000,000}{\psi_r} \right)} \right] \] 
(42)

\[ \frac{\partial \theta_w}{\partial \psi} = C(\psi) \left[ \frac{\psi^2}{\psi_r} \left( \frac{1 + \frac{\psi}{\psi_r}}{1 + \frac{1,000,000}{\psi_r}} \right) \right] \] 
(43)

\[ \frac{\partial \theta_w}{\partial a} = \frac{\theta_w \left( \frac{bc}{a} \right) \left( \frac{\psi}{\alpha} \right)^b}{e + \left( \frac{\psi}{\alpha} \right)^b} \]  
(44)

\[ \frac{\partial \theta_w}{\partial b} = \frac{-c \theta_w \left( \frac{\psi}{\alpha} \right)^b \ln \left( \frac{\psi}{\alpha} \right)}{e + \left( \frac{\psi}{\alpha} \right)^b} \]  
(45)

\[ \frac{\partial \theta_w}{\partial c} = -\theta_s \ln \left[ e + \left( \frac{\psi}{\alpha} \right)^b \right] \]  
(46)

4. Fredlund and Xing (30)

\[ \theta_w = \theta_s + \left[ \ln \left( 1 + \frac{\psi}{\psi_a} \right) \right] \] 

\[ \frac{\partial \theta_w}{\partial \psi_a} = 1 - \frac{1}{\left[ \ln \left( 1 + \frac{\psi}{\psi_a} \right) \right]^c} \]  
(47)

\[ \frac{\partial \theta_w}{\partial c} = \frac{(\theta_w - \theta_s) \left( \frac{bc}{a} \right) \left( \frac{\psi}{\alpha} \right)^b}{e + \left( \frac{\psi}{\alpha} \right)^b} \]  
(48)

\[ \frac{\partial \theta_w}{\partial a} = \frac{1}{e + \left( \frac{\psi}{\alpha} \right)^b} \ln \left[ e + \left( \frac{\psi}{\alpha} \right)^b \right] \]  
(49)

\[ \frac{\partial \theta_w}{\partial b} = \frac{-c(\theta_w - \theta_s) \left( \frac{\psi}{\alpha} \right)^b \ln \left( \frac{\psi}{\alpha} \right)}{e + \left( \frac{\psi}{\alpha} \right)^b} \]  
(50)
\[
\frac{\partial \theta_w}{\partial c} = -(\theta_w - \theta) \ln \left( \frac{e + \psi}{\psi_a} \right)
\]  
(51)

APPENDIX II. REFERENCES


